GLOBAL SOLUTIONS TO THE EIKONAL EQUATION

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Abstract. We study structural stability of smoothness of the maximal solution to the geometric eikonal equation on $(\mathbb{R}^d, G)$, $d \geq 2$. This is within the framework of order zero metrics G. For a subclass we show existence, stability as well as precise asymptotics for derivatives of the solution. These results are applicable for examples from Schrödinger operator theory.

1. Introduction and results

In this paper we investigate the existence of a smooth global solution to the geometric eikonal equation on the Riemannian manifold $(\mathbb{R}^d, G)$, $d \geq 2$, for a class of metrics G. We are interested in the so-called maximal solution $S(x)$ constructed as the geodesic distance from x to a given fixed point x_0 (taken to be $x_0 = 0$). It is well-known that for some G's this function S is smooth while for others this is not the case, in fact even $S \in C^1$ might be false. (At this point the reader may consult [Li, CC] for studies of "generalized solutions" to related Hamilton-Jacobi equations and further references.) Whence it is interesting to investigate the stability of smoothness of S under perturbation of the metric. Other issues we will study are bounds and the asymptotic behaviour of derivatives of a smooth S at infinity. This is a general mathematical problem motivated by specific applications in scattering theory, see [ACH, Ba, IS, Sk]. Most likely it is relevant for other specific problems too (possibly from control theory, geometric optics, etc) although this will not be examined in this paper.

To motivate our setup explained in further details below let us imagine a more general situation: Let us consider a complete simply connected d-dimensional manifold (M, g), $d \geq 2$, and a point $x_0 \in M$ for which the exponential map $\Phi = \exp_{x_0}(1\cdot) : TM_{x_0} \to \mathbb{R}^d$ is a diffeomorphism. Then the pullback $G = \Phi^* g$ is a metric on TM_{x_0} that can be compared with the canonical one, g_{x_0}. Introducing orthonormal coordinates we may identify $TM_{x_0} = \mathbb{R}^d$ and $g_{x_0}(y, y) = |y|^2$ (the usual Euclidean metric). Upon doing this identification the Gauss lemma (see [Ch, Theorem 1.8]) implies that $G(x)$ (considered as a matrix) has the form

$$G(x) = P + P_\perp G(x) P_\perp, \tag{1.1}$$

where P denotes, in the Dirac notation, the orthogonal projection $P = P(\hat{x}) = |\hat{x}\rangle\langle\hat{x}|$ parallel to $\hat{x} = x/|x|$ and $P_\perp = P_\perp(\hat{x}) = I - P$ the orthogonal projection onto $\{\hat{x}\}^\perp$. Note that in this picture $S(x) = |x|$ and $G(0) = I$.

Due to the above change of framework we can in principle reduce the study of a general stability problem to the one indicated above, i.e. for $(\mathbb{R}^d, G)$ only, in fact for the unperturbed metric being of the form (1.1). In this paper we shall consider families $\{G(\cdot)\}$ that are of order zero, see (1.2) and (1.3) below for precise definition. Notice that this class is naturally equipped with norms giving precise meaning to the notion of "perturbation". Our main result asserts that, under conditions, indeed upon perturbing a metric of the form (1.1) one obtains metrics, say denoted by $G_\epsilon(\cdot)$, for which the geodesic distance to the origin, $S_\epsilon(\cdot)$, is smooth (more precisely of class C^l depending on conditions) and solves the eikonal equation

$$\nabla S_\epsilon G_\epsilon^{-1} \nabla S_\epsilon = 1 \ \text{ for } x \neq 0.$$

Moreover introducing

$$s_\epsilon(x) = S_\epsilon(x)/|x| - 1 \text{ for } x \neq 0$$

we have bounds

$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} |x|^{|\alpha|} \, |\partial^\alpha s_\epsilon(x)| = o(\epsilon^0) \text{ for } |\alpha| \leq 2.$$

Depending on conditions there are somewhat similar bounds for higher order derivatives.

Finally we use the change of frame in terms of the exponential mapping for the unperturbed metric, as explained above, to solve eikonal equations of the form

$$|\nabla S_\epsilon(x)|^2 = 2(\lambda - V_\epsilon(x)) \text{ for } x \in \mathbb{R}^d \setminus \{0\},$$

where V_ϵ is given by perturbing a negative radial function (potential) obeying certain properties, see Section 7. Here the parameter $\lambda \geq 0$ plays in applications (Schrödinger operator theory) the role of energy. The conventional way af constructing solutions is by a fixed point method, cf. for example [DS, Hö1, Is]. However to our knowledge this is not doable for our examples.

This work is inspired by [Ba] which has similar results for perturbation of the special case $G = I$. As in [Ba] we shall use the geometric/variational approach to define the maximal solution to the eikonal equation, see (1.5). Many of our arguments are however very different from those of [Ba]. This came out of necessity to treat the present generality, see Remarks 1.5 3), 1.7 and 5.1 for comments on this issue.

1.1. **Conditions and main results.** Let $\mathcal{S}_d(\mathbb{R})$, $d \geq 2$, be the space of $d \times d$ symmetric matrices with components in $\mathbb{R}$ and, for $l \geq 2$, let $\mathcal{B}^l(\mathbb{R}^d)$ be the space of C^l functions $G : \mathbb{R}^d \to \mathcal{S}_d(\mathbb{R})$ such that

$$\|G\|_l = \sup\{\langle x\rangle^{|\alpha|} \, |\partial^\alpha g_{ij}(x)| : x \in \mathbb{R}^d, |\alpha| \leq l, i, j = 1, \ldots, d\} < \infty, \tag{1.2}$$

where $G(x) = (g_{ij}(x))$ and $\langle x\rangle = (1 + |x|^2)^{1/2}$. The space $\mathcal{B}^l(\mathbb{R}^d)$ endowed with the norm defined in (1.2) is a Banach space. Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ be the set of $G \in \mathcal{B}^l(\mathbb{R}^d)$ for which there are positive constants a and b satisfying

$$a|y|^2 \leq yG(x)y \leq b|y|^2, \qquad x, y \in \mathbb{R}^d. \tag{1.3}$$

The set $\mathcal{M}$ is open in $\mathcal{B}^l(\mathbb{R}^d)$, and its elements will be referred to as *metrics of order zero*. We denote by $\mathcal{H}$ the Sobolev space $(H_0^1(0,1))^d$ with the norm

$$\|h\|^2 = \langle h, h\rangle = \int_0^1 |\dot{h}(s)|^2 ds.$$

For $G \in \mathcal{M}$ we consider the energy functional $E : \mathbb{R}^d \times \mathcal{H} \to \mathbb{R}$ given, for $(x, \kappa) \in \mathbb{R}^d \times \mathcal{H}$ and $y(s) = sx + \kappa(s)$, by

$$E(x, \kappa) = \int_0^1 \dot{y}(s) G(y(s)) \dot{y}(s) ds. \tag{1.4}$$

Define for any $G \in \mathcal{M}$ a non-negative function S by

$$S^2(x) = \inf\{E(x, \kappa) : \kappa \in \mathcal{H}\}; \ x \in \mathbb{R}^d, \tag{1.5}$$

where E is as in (1.4).

Let

$$\mathcal{E}_x = \{y \in (H^1(0,1))^d : y(s) = sx + \kappa(s), \kappa \in \mathcal{H}\};\ x \in \mathbb{R}^d. \tag{1.6}$$

In agreement with usual convention [Mi], we say that $\gamma \in \mathcal{E}_x$ is a *geodesic of* G *emanating from* 0 *with value* x *at time one* if $\partial_\kappa E(x,\kappa) = 0$ as an element of the dual space $\mathcal{H}'$ of $\mathcal{H}$. Geodesics can emanate from other points in $\mathbb{R}^d$. The general definition of a *geodesic of* G ([Ch, Mi]) may be taken to be an orbit $s \to \gamma(s) \in \mathbb{R}^d$ solving a certain second order differential equation, see (2.12). These solutions can be extended to globally defined solutions, cf. the Hopf Rinow theorem [Ch, Theorem 1.10]. By the Riesz lemma we can identify $\mathcal{H}'$ and $\mathcal{H}$, as well as the set of bounded quadratic forms on $\mathcal{H}$ by the set of bounded self-adjoint operators on $\mathcal{H}$.

We shall consider two sets of metrics of order zero. The first one is the following.

Definition 1.1. Let $\mathcal{U}$ be the subclass of order zero metrics G that satisfy:
1) For every $x \in \mathbb{R}^d$ the metric G has a unique geodesic $\gamma_x(s) = sx + \kappa_x(s)$ in $\mathcal{E}_x$.
2) There exists $c > 0$ independent of $x \in \mathbb{R}^d$ such that if $\kappa = \kappa_x$ is given as in 1)

$$\langle \partial_\kappa^2 E(x,\kappa)h, h\rangle \geq c\|h\|^2, \qquad h \in \mathcal{H}. \tag{1.7}$$

Remarks. 1) The Hessian appearing in (1.7) is given by

$$\langle \partial_\kappa^2 E(x,\kappa)h_1, h_2\rangle = \int_0^1 (2\dot{h}_1 G\dot{h}_2 + 2\dot{y}\nabla G \cdot h_1\dot{h}_2 + 2\dot{y}\nabla G \cdot h_2\dot{h}_1 + \dot{y}(\nabla^2 G; h_1, h_2)\dot{y})ds; \tag{1.8}$$

here $\nabla G \cdot h$ stands for the matrix $(\nabla g_{ij} \cdot h)$. Let us for completeness of presentation remark that for all $G \in \mathcal{M}$ satisfying 1) there exists $C > 0$ independent of $x \in \mathbb{R}^d$ such that

$$|\langle \partial_\kappa^2 E(x,\kappa)h, h\rangle| \leq C\|h\|^2, \qquad h \in \mathcal{H}.$$

(This follows readily from (1.8), (2.5) and (3.11a).)

2) For all $G \in \mathcal{U}$ the corresponding exponential map $\Phi = \exp_0(1\cdot) : T\mathbb{R}^d_0 \to \mathbb{R}^d$ is a diffeomorphism, cf. [Ch, Theorem 2.16] and [Mi, Theorem 14.1].

Proposition 1.2. *Let* $G \in \mathcal{U}$. *The non-negative function* S *defined by* (1.5) *is of class* C^l *on* $\mathbb{R}^d \setminus \{0\}$ *and satisfies the eikonal equation*

$$\nabla S G^{-1} \nabla S = 1 \ \text{ for } x \neq 0. \tag{1.9}$$

Furthermore, there exists $C > 0$ *such that*

$$\sup_{|x|\geq 1} \langle x\rangle^{\min(|\alpha|-1,\,|\alpha|/2)}\, |\partial^\alpha S(x)| \leq C \text{ for all } |\alpha| \leq l. \tag{1.10}$$

Remark. One might suspect that (1.10) can be replaced by the stronger bounds

$$\sup_{|x|\geq 1} \langle x\rangle^{|\alpha|-1}\, |\partial^\alpha S(x)| \leq C \text{ for all } |\alpha| \leq l. \tag{1.11}$$

In general this is an open problem. On the other hand (1.10) appears natural if the class of metrics is enlarged by replacing (1.2) by

$$\widetilde{\|G\|_l} = \sup\{\langle x\rangle^{\min(|\alpha|,1+|\alpha|/2)}\, |\partial^\alpha g_{ij}(x)| : x \in \mathbb{R}^d, |\alpha| \leq l, i,j = 1,\ldots,d\} < \infty.$$

In this situation indeed the analogue version of Proposition 1.2 holds true (showned by the same proof).

Our second set of metrics of order zero is given as follows.

Definition 1.3. Let $\mathcal{O}$ be the subset of order zero metrics G obeying:

1) For $x \neq 0$
$$G(x) = P + P_\perp G(x) P_\perp, \tag{1.12}$$
where P denotes, in the Dirac notation, the orthogonal projection $P = P(\omega) = |\omega\rangle\langle\omega|$ parallel to $\omega = \hat{x} = x/|x|$ and $P_\perp = P_\perp(\omega) = I - P$ the orthogonal projection onto $\{\omega\}^\perp$.
2) There exists $\bar{c} > 0$ such that for $x \neq 0$
$$P_\perp\big(G(x) + 2^{-1} x \cdot \nabla G(x)\big) P_\perp \geq \bar{c} P_\perp G(x) P_\perp. \tag{1.13}$$

Note that since G is continuous at $x = 0$ Definition 1.3 1) implies that $G(0) = I$. Moreover it follows from (1.13) that $\bar{c} \in]0, 1]$. The simplest example of a metric of order zero satisfying (1.12) and (1.13) is $G = I$, the $d \times d$ identity matrix. In Section 7 we provide other examples of metrics of order zero satisfying (1.12) and (1.13). We may refer to (1.13) as a *convexity property* since, given the orthogonal decomposition (1.12) the estimate is equivalent to the geometric Hessian bound
$$\nabla^2 S(x)^2 \geq 2\bar{c} g(x). \tag{1.14}$$
Here we use the conventional metric notation g rather than the matrix notation G and (with (1.12)) $S(x) = |x|$. Written in this way the condition (1.13) clearly becomes *geometrically invariant* which a priori is a desirable property. However in computations we shall only use (1.13).

In terms of the non-negative function S given by (1.5) (for any $G \in \mathcal{M}$) let
$$s(x) = S(x)/|x| - 1 \text{ for } x \neq 0.$$
Our first main result is:

Theorem 1.4. *Let $\mathcal{U}, \mathcal{O} \subseteq \mathcal{M}$ be given by Definitions 1.1 and 1.3. There exists a neighbourhood $\widetilde{\mathcal{O}} \subseteq \mathcal{M}$ of $\mathcal{O}$ such that:*

i) $\widetilde{\mathcal{O}} \subseteq \mathcal{U}$; that is the set $\mathcal{O}$ is a subset of the interior of $\mathcal{U}$.
ii) Let $G \in \mathcal{O}$ be given. Then there exist $\epsilon_0, C > 0$ such that for all $\tilde{G} \in \mathcal{M}$ with $\|\tilde{G} - G\|_l \leq \epsilon_0$ not only $\tilde{G} \in \widetilde{\mathcal{O}}$ but also
$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} |s(x)| \leq C \|\tilde{G} - G\|_l, \tag{1.15a}$$
$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} |x|\, |\partial^\alpha s(x)| \leq C \|\tilde{G} - G\|_l^{3/4} \text{ for } |\alpha| = 1, \tag{1.15b}$$
$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} |x|^2\, |\partial^\alpha s(x)| \leq C \|\tilde{G} - G\|_l^{1/2} \text{ for } |\alpha| = 2. \tag{1.15c}$$

Remarks 1.5. 1) One might presume that $\mathcal{U}$ is open in $\mathcal{M}$. However if true at all this is a hard problem. Similarly (seemingly a softer problem) one might presume that if we drop the condition (1.13) of the Definition 1.3 and call this bigger class $\mathcal{O}_1$, then $\mathcal{O}_1$ is a subset of the interior of U. (Note that $\mathcal{O}_1 \subseteq U$ due to Lemma 4.1.). Even this problem appears to be difficult. In our approach we use (1.13) crucially to obtain good control of perturbed geodesics uniformly in x. Given the lower bound in (1.3) the condition (1.13) is a somewhat weak assumption.

2) The two constants $\epsilon_0, C > 0$ can be taken as locally bounded functions of $(\|G\|_l, a, \bar{c}) \in \mathbb{R}^3_+$ where the entries $\|G\|_l$, a, and $\bar{c}$ are defined by (1.2), (1.3) and (1.13) for the metric G, respectively. Obviously this statement for $l \geq 3$ follows from the assertion for $l = 2$.
3) For perturbations of the Euclidean metric $G = I$ one can replace the powers to the right in (1.15a)–(1.15c) by the more natural factor $\|\tilde{G} - G\|_l$, see [Ba]. However the techniques of [Ba] are not applicable in our case, see Remark 5.1 for some elaboration at this point. We do not know if this improvement is possible in our more general case.

The estimate (1.15b) is a consequence, by interpolation, of the bounds (1.15a) and (1.15c), cf. [Hö2, proof of Lemma 7.7.2]. We shall use (1.15a) and the following weaker version of (1.15b) in the proof of (1.15c):

$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} |x|\, |\partial^\alpha s(x)| \leq C \|\tilde{G} - G\|_l^{1/2} \text{ for } |\alpha| = 1. \tag{1.16}$$

Our second main result supplements Theorem 1.4 ii). It reads

Theorem 1.6. *Suppose $l \geq 3$. Let $G \in \mathcal{O}$ and $r > 0$ be given. Then there exist $\epsilon_0, C > 0$ such that for all $\tilde{G} \in \mathcal{M}$ with $\|\tilde{G} - G\|_l \leq \epsilon_0$ and $|\alpha| \leq l$*

$$\sup_{|x| \geq r} \langle x \rangle^{|\alpha|-1} |\partial^\alpha S(x)| \leq C, \tag{1.17a}$$

$$\sup \langle x \rangle^{|\alpha|-2} \left|\partial^\alpha S^2(x)\right| \leq C. \tag{1.17b}$$

Remark 1.7. If $l \geq 4$ we can use Theorems 1.4 and 1.6 and interpolation to show that

$$\sup_{|x| \geq r} |x|^{|\alpha|} |\partial^\alpha s(x)| \leq C_r \|\tilde{G} - G\|_l^{2^{1-|\alpha|}} \text{ for } 2 \leq |\alpha| \leq l - 1. \tag{1.18}$$

A slightly improved bound for $|\alpha| = 3$ was proved in [Ba] (in the setting of [Ba]) under the assumption that $l = 3$. We do not need estimates like (1.18) for $|\alpha| > 2$ in our applications [IS, Sk]. On the other hand Theorem 1.6 for $l = 3$ is indeed important in these applications.

This paper is organized as follows: In Section 2 we study the minimization problem (1.5). Using standard arguments we show the existence of a minimizer and some basic properties of any such minimizer. In Section 3 we show Proposition 1.2. The proof is based on the implicit function theorem, and the somewhat lengthy scheme for proving the bounds (1.10) is used again in Section 6 to establish the improved bounds of Theorem 1.6. The proof of Theorem 1.4 i), given in Section 4, is based on an analysis yielding dynamical control of perturbed geodesics. We obtain sufficient control to be able to deduce the uniqueness of the energy minimizer from a result from global analysis. In Section 5 we show Theorem 1.4 ii) by using various explicit computations in combination with results from Section 4. The proof of Theorem 1.6, given in Section 6, is based on functional analysis arguments for Hardy spaces taylored to the problem in hand. The first part of the proof is devoted entirely to setting this up abstractly. The second part is devoted to verification of conditions, and as indicated above, this involves a scheme from Section 3. In Section 7 we present examples from Schrödinger operator theory.

2. The minimization problem

In this section we study some basic properties for metrics $G \in \mathcal{M}$. Since $S(x) = |x|$ when $G(x) = I$ then, from (1.3) and (1.5), we have

$$a|x|^2 \le S^2(x) \le b|x|^2, \tag{2.1}$$

for all $x \in \mathbb{R}^d$.

Lemma 2.1. *Let $G \in \mathcal{M}$ and E as in* (1.4)*. Then for every $x \in \mathbb{R}^d$ there exists $\kappa \in \mathcal{H}$ such that*

$$S^2(x) = E(x, \kappa). \tag{2.2}$$

Moreover, if γ is a geodesic of G emanating from 0 *with value x at time one then*

$$\dot\gamma(s)G(\gamma(s))\dot\gamma(s) = \int_0^1 \dot\gamma(t)G(\gamma(t))\dot\gamma(t)dt, \qquad s \in [0,1]. \tag{2.3}$$

In particular, if γ is a minimizer of (1.4) *then for all $s \in [0,1]$ we have*

$$\dot\gamma(s)G(\gamma(s))\dot\gamma(s) = S^2(x), \tag{2.4}$$

and

$$\frac{a}{b}|x|^2 \le |\dot\gamma(s)|^2 \le \frac{b}{a}|x|^2 \qquad \text{and} \qquad \frac{a}{b}|sx|^2 \le |\gamma(s)|^2 \le \frac{b}{a}|sx|^2. \tag{2.5}$$

Proof. **Existence of a minimizer.** To establish the existence of a minimizer it suffices to show that, for any fixed $x \in \mathbb{R}^d$, the functional $E(x,\cdot)$ is weakly lower semicontinuous on $\mathcal{H}$, and that

$$\lim_{\|\kappa\|\to\infty} E(x,\kappa) = \infty. \tag{2.6}$$

Note that (2.6) follows from the estimate, $y(s) := sx + \kappa(s) \in \mathcal{E}_x$,

$$E(x,\kappa) = \int_0^1 \dot y(s)G(y(s))\dot y(s)ds \ge a\int_0^1 |\dot y(s)|^2 ds.$$

To prove the weak lower semicontinuity of E in the second variable we let $\{\kappa_n\}$ be a sequence in $\mathcal{H}$ that converges weakly to $\kappa \in \mathcal{H}$ and write $y(s) = sx + \kappa(s)$ and $y_n(s) = sx + \kappa_n(s)$. Then

$$\begin{aligned} E(x,\kappa_n) - E(x,\kappa) = &\int_0^1 (\dot y_n - \dot y)G(y)(\dot y_n - \dot y)ds - 2\int_0^1 \dot y G(y)(\dot y - \dot y_n)ds \\ &- \int_0^1 \dot y_n(G(y) - G(y_n))\dot y_n ds. \end{aligned}$$

Since the first integral is non-negative, the second goes to zero as $n \to \infty$ (because $y_n - y$ converges weekly to zero) and the third integral goes to zero too (by the compact embedding of $\mathcal{H}$ into $(C(0,1))^d$) we find that

$$\liminf_{n\to\infty} E(x,\kappa_n) \ge E(x,\kappa).$$

The conservation law. Since G is of class C^l so is $E(x,\kappa)$ and for all $h \in \mathcal{H}$ we have

$$\langle \partial_\kappa E(x,\kappa), h\rangle = \int_0^1 [\dot y \nabla G \cdot h \dot y + 2\dot h G \dot y]ds. \tag{2.7}$$

Moreover, if γ is a geodesic of G then for all $h \in \mathcal{H}$

$$\int_0^1 (\dot\gamma \nabla G \cdot h\dot\gamma + 2\dot h G \dot\gamma) ds = 0. \tag{2.8}$$

Integrating by parts we obtain

$$\int_0^1 [\dot\gamma \nabla G \cdot h\dot\gamma - 2h(\nabla G \cdot \dot\gamma\dot\gamma + G\ddot\gamma)] dt = 0. \tag{2.9}$$

Thus, in the space $\mathcal{H}'$ we have

$$2e_k G\ddot\gamma = \dot\gamma \nabla G \cdot e_k \dot\gamma - 2e_k \nabla G \cdot \dot\gamma\dot\gamma \tag{2.10}$$

for each $k = 1, \dots, d$, where e_k is the k^{th} element of the standard basis of $\mathbb{R}^d$; $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, etc. This implies that

$$\ddot\gamma = \tfrac{1}{2} G^{-1} w, \tag{2.11a}$$

where the k^{th} component w_k of the function $w : [0,1] \to \mathbb{R}^d$ is given by

$$w_k = \dot\gamma \nabla G \cdot e_k \dot\gamma - 2e_k \nabla G \cdot \dot\gamma\dot\gamma. \tag{2.11b}$$

Letting $e_\bullet$ denote the ordered basis $e_1, \dots, e_d$ of $\mathbb{R}^d$ we can write (2.11a) and (2.11b) more compactly as

$$\ddot\gamma = \ddot\kappa = 2^{-1} G^{-1} \dot\gamma \nabla G \cdot e_\bullet \dot\gamma - G^{-1} \nabla G \cdot \dot\gamma\dot\gamma. \tag{2.12}$$

Multiplying (2.10) by $\dot\gamma_k$ and adding over k yields

$$\dot\gamma \nabla G \cdot \dot\gamma\dot\gamma - 2\dot\gamma \nabla G \cdot \dot\gamma\dot\gamma - 2\dot\gamma G \ddot\gamma = 0,$$

from which we obtain

$$\frac{d}{ds}\left(\dot\gamma(s) G(\gamma(s)) \dot\gamma(s)\right) = 0, \qquad s \in [0,1], \tag{2.13}$$

and therefore (2.3) and (2.4) follow.

The estimates. Suppose γ is a minimizer of (1.4). Combining (1.3), (2.1) and (2.4) we obtain the estimates for $|\dot\gamma(s)|$ stated in (2.5).

Furthermore, using the upper bound for $|\dot\gamma(s)|$ we find that

$$|\gamma(s)| \le \int_0^s |\dot\gamma(\tau)| d\tau \le \sqrt{\frac{b}{a}} |sx|, \qquad \text{for all } s \in [0,1].$$

Suppose now that for some $s_1 \in (0,1]$ we have $|\gamma(s_1)|^2 < (a/b)|s_1 x|^2$, then defining $y(s) = (s/s_1)\gamma(s_1)$ for $0 \le s \le s_1$ and $y(s) = \gamma(s)$ for $s_1 \le s \le 1$ we have

$$E(y) = \int_0^{s_1} s_1^{-2} \gamma(s_1) G(y(s)) \gamma(s_1) ds + \int_{s_1}^1 \dot\gamma(s) G(\gamma(s)) \dot\gamma(s) ds.$$

Thus, using (1.3), (2.1), (2.4), and our assumption on s_1 we find that

$$E(y) \le \frac{b|\gamma(s_1)|^2}{s_1} + (1 - s_1) S^2(x) < a|x|^2 s_1 + (1 - s_1) S^2(x) \le S^2(x),$$

which is impossible by the choice of γ. Thus (2.5) holds and the proof is complete. □

3. Proof of Proposition 1.2

Clearly Proposition 1.2 follows from

Proposition 3.1. *Let $G \in \mathcal{U}$ and write the unique geodesic for G with endpoint x as $\gamma_x(s) = sx + \kappa_x(s)$.*

i) The map $\mathbb{R}^d \ni x \to \kappa_x \in \mathcal{H}$ is of class C^{l-1}.

ii) The non-negative function S in (1.5) obeys

$$S^2(x) = \int_0^1 \dot\gamma_x(s) G(\gamma_x(s)) \dot\gamma_x(s) ds, \tag{3.1}$$

and it is a C^l solution to the eikonal equation (1.9).

iii) There are bounds

$$\|\partial^\alpha \kappa\| \le C \langle x \rangle^{1/2 - \min(|\alpha|-1/2,\, |\alpha|/2)} \text{ for all } |\alpha| \le l-1. \tag{3.2}$$

iv) For any $r > 0$

$$\sup_{|x|\ge r} \langle x \rangle^{\min(|\alpha|-1,\, |\alpha|/2)} |\partial^\alpha S(x)| \le C_r \text{ for all } |\alpha| \le l. \tag{3.3}$$

v) In iii) and iv) the constants C and C_r can be taken as locally bounded functions of $(\|G\|_l, a, c) \in \mathbb{R}_+^3$ and $(\|G\|_l, a, c, r) \in \mathbb{R}_+^4$, respectively. Here the entries $\|G\|_l$, a and c are defined by (1.2), (1.3) and (1.7), respectively.

Proof. **Re i).** The statement follows from our assumption that $G \in \mathcal{U}$, the representation (2.7) and the implicit function theorem. Note that indeed the classical implicit function theorem given for example in [De, Theorem 15.1] (see possibly also [Ir, Theorem C.7]) can be applied to the equation $\partial_\kappa E(x,\kappa) = 0$ in a small neighborhood of any fixed $(x_0, \kappa_{x_0}) \in (\mathbb{R}^d, \mathcal{H})$. The unique solution is a map $x \to \tilde\kappa_x$ of class C^{l-1} from a neighborhood of $x_0 \in \mathbb{R}^d$ to $\mathcal{H}$. Since the corresponding geodesic $\tilde\gamma_x$, $\tilde\gamma_x(s) = sx + \tilde\kappa_x$, by our uniqueness assumption coincides with γ_x we deduce that $\kappa_x = \tilde\kappa_x$. Whence also $x \to \kappa_x$ is of class C^{l-1}.

Re ii). Clearly (3.1) is a consequence of Lemma 2.1 and the uniqueness of geodesics. It follows from i) and (3.1) that S^2 is of class C^{l-1} on $\mathbb{R}^d$. In particular, since $S(x) > 0$ for $x \in \mathbb{R}^d \setminus \{0\}$, S is of class C^1 on $\mathbb{R}^d \setminus \{0\}$. If we write γ instead of γ_x in (3.1), then using (2.9), (2.10), and the fact that $\partial\gamma/\partial x_k = se_k + \partial\kappa/\partial x_k$, with $\kappa \in \mathcal{H}$, we have

$$\begin{aligned} 2S(x)\frac{\partial S(x)}{\partial x_k} &= \int_0^1 \left(2\frac{\partial \dot\gamma}{\partial x_k} G\dot\gamma + \dot\gamma \nabla G \cdot \frac{\partial \gamma}{\partial x_k}\dot\gamma \right) ds \\ &= \int_0^1 (2e_k G\dot\gamma + s\dot\gamma\nabla G \cdot e_k \dot\gamma) ds \\ &= \int_0^1 ((2se_k G\dot\gamma)^{\cdot} + s(\dot\gamma\nabla G\cdot e_k\dot\gamma - (2e_k G\dot\gamma)^{\cdot})\, ds \\ &= 2e_k G(x)\dot\gamma(1). \end{aligned}$$

Thus

$$G(x)\dot\gamma(1) = S(x)\nabla S(x), \tag{3.4}$$

and using (2.3) we obtain

$$S^2(x) = \dot\gamma(1) G(\gamma(1)) \dot\gamma(1) = S^2(x) \nabla S(x) G(x)^{-1} \nabla S(x),$$

from which (1.9) follows.

It remains to show that S is of class C^l on $\mathbb{R}^d \setminus \{0\}$: From (3.4) we obtain

$$\nabla S(x) = S(x)^{-1} G(x)(x + \dot{\kappa}_x(1)). \tag{3.5}$$

Whence it suffices to show that $\dot{\kappa}_x(1)$ is of class C^{l-1}. For that let us note the representation

$$\dot{\kappa}(1) = 2 \int_{1/2}^{1} \Big(\dot{\kappa}(s) + \int_s^1 \ddot{\kappa}(\sigma) d\sigma\Big) ds. \tag{3.6}$$

The right hand side of (3.6) is indeed of class C^{l-1}. Note that in fact it follows from (3.6) that

$$\partial_x^\alpha \dot{\kappa}(1) = 2 \int_{1/2}^{1} \Big(\partial_x^\alpha \dot{\kappa}(s) + \int_s^1 \partial_x^\alpha \ddot{\kappa}(\sigma) d\sigma\Big) ds \text{ for } |\alpha| \le l-1, \tag{3.7}$$

where the quantities $\partial_x^\alpha \dot{\kappa}$ and $\partial_x^\alpha \ddot{\kappa}$ are well-defined $\big(L^1(0,1)\big)^d$−valued function, cf. i) and (2.12).

Re iii). It suffices to show the bounds for $|x| \ge 1$. We shall proceed by induction in $|\alpha|$. Note that (3.2) for $|\alpha| = 0$ only follows from Lemma 2.1 and the representation $\kappa(s) = \gamma(s) - sx$. Suppose we know the bounds for $|\alpha| \le n-1$ then we need to show these for $|\alpha| = n$. So let α with $|\alpha| = n$ be given.

By repeated differentation of the defining equation $\langle \partial_\kappa E(x,\kappa), h\rangle = 0$ (for any $h \in \mathcal{H}$) we obtain that

$$-\langle \partial_\kappa^2 E(x,\kappa) \partial^\alpha \kappa, h\rangle$$

is a sum of terms each one either of the form

$$\langle \partial_x^\zeta \partial_\kappa E(x,\kappa), h\rangle = (\partial_x^\zeta \partial_\kappa^{1+k} E(x,\kappa); h); |\zeta| = n,\ k = 0, \tag{3.8a}$$

or of the form, for $k = 1, \dots, n$,

$$(\partial_x^\zeta \partial_\kappa^{1+k} E(x,\kappa); \partial^{\beta_1}\kappa, \dots, \partial^{\beta_k}\kappa, h); \tag{3.8b}$$

$$|\zeta| + \sum_{j=1}^{k} |\beta_j| = |\alpha| = n \text{ and } 1 \le |\beta_j| \le n-1.$$

Due to (1.7) it suffices to bound the expressions in (3.8a) and (3.8b) as

$$|\cdots| \le C\langle x\rangle^{1/2 - n/2}. \tag{3.9}$$

From (2.7) we see that

$$(\partial_x^\zeta \partial_\kappa^{1+k} E(x,\kappa); h_1, \dots, h_k, h_{k+1})$$

is an integral of a sum of $(k+1)$-tensors in $g_1(s), \dots, g_{k+1}(s)$ where for each $j \le k+1$ either $g_j(s)$ is a component of $h_j(s)$ or $g_j(s)$ is a component of $\dot{h}_j(s)$. At most two factors have a "dot superscript", and if we include factors of components of $\dot{\gamma}(s)$ and factors of components of $\partial_{x_i} \frac{d}{ds}\{sx\} = e_i$, $i = 1, \dots, d$, the total number of such factors is for all tensors exactly two. Whence we are motivated to group the terms into three types that will be considered separately below:

A) There are no factors of components of $\dot{\gamma}(s)$.
B) There is one factor of a component of $\dot{\gamma}(s)$.
C) There are two factors of components of $\dot{\gamma}(s)$.

Clearly these tensors involve factors of components of $\partial^\eta G(\gamma(s))$ also. These are estimated as

$$|\partial^\eta g_{ij}| \leq C|sx|^{-\sigma \min(|\eta|,\, 1+|\eta|/2)};\ \sigma \in [0,1], \tag{3.10}$$

due to Lemma 2.1. The singular power of s in (3.10) (depending on the σ at our disposal) needs to be factorized into factors some of which need to be distributed to factors of components of $h_j(s)$ (if such components appear) and then "removed" either by the Hardy inequality (removing a factor s^{-1})

$$\int_0^1 s^{-2}|\tilde h(s)|^2\,\mathrm{d}s \leq 4\|\tilde h\|^2, \tag{3.11a}$$

or by the estimate (removing a factor $s^{-1/2}$)

$$|\tilde h(s)| \leq \sqrt{s}\|\tilde h\|. \tag{3.11b}$$

Another factor of the singular power of s in (3.10) combines with factors of components of $\partial_{x_i}(\gamma(s) - \kappa(s)) = se_i$, $i = 1, \dots, d$.

To summarize we need to look at the expressions (3.8a) and (3.8b). Let $h_j = \partial^{\beta_j}\kappa$ for $j \leq k$ and $h_{k+1} = h$. After doing a complete expansion into terms (using the product rule for differentiation) we need to bound each resulting expression say $(F_{\zeta,k}; h_1, \dots, h_{k+1})$. Recall from the above discussion that any such term is an integral of a $(k+1)$-tensorial expression; the j'th factor is either a component of $h_j(s)$ or a component of $\dot h_j(s)$. The treatment of these terms is divided into various cases. For simplicity we assume below that $k \geq 1$. For $k = 0$ we can argue similarly (although the treatment for $k = 0$ is simpler).

Case A): Notice that we need $|\eta| = |\zeta| + k - 1$ in (3.10). We distinguish between the following cases: **Ai)** There occur a component of $\dot h_i(s)$ and a component of $\dot h_j(s)$ (for some $i \neq j$). **Aii)** Exactly one factor of component of $\dot h_j(s)$ occurs. **Aiii)** There is no factor of component of $\dot h_j(s)$.

Case Ai): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^\zeta \partial_{\kappa(s)}^\omega G(sx + \kappa(s))$; $|\omega| = k - 1$. We choose in (3.10) $\sigma \in [0,1]$ such that with the given value of $|\eta|$

$$\sigma \min(|\eta|,\, 1 + |\eta|/2) = |\eta|/2 =: K. \tag{3.12}$$

Upon using the pointwise bound (3.11b) for $k-1$ factors, the pointwise estimate

$$s^{-K}s^{|\zeta|}s^{(k-1)/2} \leq 1$$

and the Cauchy Schwarz inequality we obtain the bound

$$|(F_{\zeta,k}; h_1, \dots, h_{k+1})| \leq C|x|^{-K}\prod_{m=1}^{k+1}\|h_m\|. \tag{3.13}$$

By the induction hypothesis

$$\prod_{m=1}^{k}\|h_m\| \leq C\langle x\rangle^{k/2 - \sum|\beta_j|/2} = C\langle x\rangle^{k/2-(n-|\zeta|)/2},$$

which together with (3.13) yields

$$|(F_{\zeta,k}; h_1, \dots, h_{k+1})| \leq C\langle x\rangle^{-K}\langle x\rangle^{k/2-(n-|\zeta|)/2}\|h\| = C\langle x\rangle^{1/2-n/2}\|h\|. \tag{3.14}$$

Case Aii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_1|}\partial_x^{\zeta_1}\partial_{\kappa(s)}^{\omega} G(sx+\kappa(s))$; $|\zeta_1| = |\zeta|-1,\ |\omega| = k$. We choose σ as in (3.12). Upon using the bound (3.11a) for one factor and the pointwise bound (3.11b) for $k-1$ factors we proceed as in the previous case using now that

$$s^{-|\eta|/2}s^{|\zeta_1|}s^{1+(k-1)/2} \le 1.$$

Case Aiii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_2|}\partial_x^{\zeta_2}\partial_{\kappa(s)}^{\omega} G(sx+\kappa(s))$; $|\zeta_2| = |\zeta|-2,\ |\omega| = k+1$. We choose σ as in (3.12). Upon using the bound (3.11a) for two factors and the pointwise bound (3.11b) for $k-1$ factors we precced as in the first case using now that

$$s^{-|\eta|/2}s^{|\zeta_2|}s^{2+(k-1)/2} \le 1.$$

Case B): We need $|\eta| = |\zeta| + k$ in (3.10). We distinguish between the following cases: **Bi)** Exactly one factor of component of $\dot{h}_j(s)$ occurs. **Bii)** There is no factor of component of $\dot{h}_j(s)$.

Case Bi): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^{\zeta}\partial_{\kappa(s)}^{\omega} G(sx+\kappa(s))$; $|\omega| = k$. We choose in (3.10) $\sigma\in[0,1]$ such that with the given value of $|\eta|$

$$\sigma \min(|\eta|,\, 1+|\eta|/2) = 1/2 + |\eta|/2 =: K. \tag{3.15}$$

Upon using the bound (3.11a) for one factor and the pointwise bound (3.11b) for $k-1$ factors, the pointwise estimate

$$s^{-K}s^{|\zeta|}s^{1+(k-1)/2} \le 1$$

and the Cauchy Schwarz inequality we obtain the bound

$$|(F_{\zeta,k}; h_1,\dots,h_{k+1})| \le C|x|^1|x|^{-K}\prod_{m=1}^{k+1}\|h_m\|. \tag{3.16}$$

(The factor $|x|^1$ comes from bounding $|\dot\gamma(s)| \le C|x|$, cf. Lemma 2.1.) By the induction hypothesis

$$\prod_{m=1}^{k}\|h_m\| \le C\langle x\rangle^{k/2-(n-|\zeta|)/2},$$

yielding

$$|(F_{\zeta,k}; h_1,\dots,h_{k+1})| \le C\langle x\rangle^{1-K}\langle x\rangle^{k/2-(n-|\zeta|)/2}\|h\| = C\langle x\rangle^{1/2-n/2}\|h\|. \tag{3.17}$$

Case Bii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_1|}\partial_x^{\zeta_1}\partial_{\kappa(s)}^{\omega} G(sx+\kappa(s))$; $|\zeta_1| = |\zeta|-1,\ |\omega| = k+1$. We choose σ as in (3.15). Upon using the bound (3.11a) for two factors and the pointwise bound (3.11b) for $k-1$ factors we proceed as in case Bi) using now that

$$s^{-K}s^{|\zeta_1|}s^{2+(k-1)/2} \le 1.$$

For the case C) we need (3.10) with $|\eta| = |\zeta| + k + 1$.

Case C): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^{\zeta}\partial_{\kappa(s)}^{\omega} G(sx+\kappa(s))$; $|\omega| = k+1$. We choose in (3.10) $\sigma\in[0,1]$ such that with the given value of $|\eta|$

$$\sigma \min(|\eta|,\, 1+|\eta|/2) = 1 + |\eta|/2 =: K. \tag{3.18}$$

Upon using the bound (3.11a) for two factors and the pointwise bound (3.11b) for $k-1$ factors, the pointwise estimate

$$s^{-K}s^{|\zeta|}s^{2+(k-1)/2} \le 1$$

and the Cauchy Schwarz inequality we obtain the bound

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})|\le C|x|^2|x|^{-K}\prod_{m=1}^{k+1}\|h_m\|. \tag{3.19}$$

By the induction hypothesis

$$\prod_{m=1}^{k}\|h_m\|\le C\langle x\rangle^{k/2-(n-|\zeta|)/2},$$

yielding

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})|\le C\langle x\rangle^{2-K}\langle x\rangle^{k/2-(n-|\zeta|)/2}\|h\|=C\langle x\rangle^{1/2-n/2}\|h\|. \tag{3.20}$$

Re iv). The proof is by induction in $|\alpha|$ and based on (3.5), (3.7), (2.12) and iii). We notice that bound

$$|\partial^\alpha\dot\kappa(1)|\le C\langle x\rangle^{1/2-\min(|\alpha|-1/2,\,|\alpha|/2)}\text{ for all }|\alpha|\le l-1. \tag{3.21}$$

Indeed by repeated differentiation of (2.12) it suffices, due to (3.7), to bound the $L^1(1/2,1)$-norm of quantities of the form

$$\partial^\eta g_{mn}\partial^{\beta_1}\dot\gamma_{j_1}(s)\,\partial^{\beta_2}\dot\gamma_{j_2}(s)\,\partial^{\omega_1}\gamma_{i_1}(s)\cdots\partial^{\omega_k}\gamma_{i_k}(s);$$
$$|\eta|=k+1,\ |\omega_i|\ge 1,\ \sum_{j=1,2}|\beta_j|+\sum_{i=1,\dots,k}|\omega_i|=|\alpha|.$$

Using the bounds (3.10) and iii) we obtain that

$$\|\partial^\alpha\ddot\kappa\|_{L^1(1/2,1)}\le C\langle x\rangle^{1/2-\min(|\alpha|-1/2,\,|\alpha|/2)},$$

yielding (3.21).

Now we apply the product rule to (3.5) noticing that also derivatives of the last factor $\dot\gamma(1)=x+\dot\kappa_x(1)$ obey the bounds (3.21). Using the fact that $n\to\min(n-2,(n-1)/2)$ is concave and the induction hypothesis it follows that

$$|\partial^\eta S(x)^{-1}|\le C\langle x\rangle^{-\min(|\eta|-1,\,|\eta|/2)-2}.$$

Since also

$$|\partial^\eta g_{mn}(x)|\le C\langle x\rangle^{-\min(|\eta|,\,1+|\eta|/2)},$$

indeed the product rule (and a little bookkeeping effort) completes the induction argument.

Re v). This is obvious from the above proofs.

□

4. Proof of Theorem 1.4 i)

We embark on proving the first assertion of Theorem 1.4.

4.1. **Unperturbed case.** Let $\mathcal{O}_1$ be the subset of order zero metrics obeying Definition 1.3 1) and let $\mathcal{U}$ be given as in Definition 1.1.

Lemma 4.1. $\mathcal{O}_1 \subseteq \mathcal{U}$.

Proof. Let $G \in \mathcal{O}_1$ be given.

Re Definition 1.1 1). A short calculation, using (1.12), gives that for all $h \in \mathbb{R}^d$

$$\nabla G \cdot h = \nabla P \cdot h + P_\perp \nabla G \cdot h P_\perp + \nabla P_\perp \cdot h G P_\perp + P_\perp G \nabla P_\perp \cdot h; \tag{4.1a}$$

$$\nabla P \cdot h = |\frac{P_\perp h}{|x|}\rangle\langle\omega| + |\omega\rangle\langle\frac{P_\perp h}{|x|}| \qquad \text{and} \qquad \nabla P_\perp \cdot h = -\nabla P \cdot h. \tag{4.1b}$$

Now, using (4.1a) and (4.1b) it can easily be verified that $\gamma(s) = sv$ is the solution to (2.12) that satisfies $\gamma(0) = 0$, and $\dot\gamma(0) = v$, for any given $v \in \mathbb{R}^d$. By standard uniqueness of the solution to an initial value problem for an ordinary differential equation this γ is the unique solution to (2.12) that satisfies $\gamma(0) = 0$, and $\dot\gamma(0) = v$. Since $\gamma \in \mathcal{E}_x$ is a geodesic of G if and only if γ satisfies (2.12), $\gamma(0) = 0$, and $\gamma(1) = x$, then $\gamma(s) = sx$ for $s \in [0,1]$, and thus G satisfies Definition 1.1 1).

Re Definition 1.1 2). We will use the representation (1.8). Since for all $x \in \mathbb{R}^d$ the geodesic is $\gamma(s) = sx$, a not very short but elementary calculation, using (4.1a), (4.1b), and the shorthand notation $\dot\gamma = d\gamma/ds$, $h_\perp = P_\perp h$ and $\gamma \cdot \omega = \langle \gamma, \omega\rangle$, yields

$$\begin{aligned}
4&\int_0^1 \dot\gamma \nabla G \cdot h \dot h ds \\
&= 4\int_0^1 \left\{ \frac{h_\perp \cdot \dot h}{|\gamma|}\dot\gamma\cdot\omega - \frac{h_\perp G \dot h_\perp}{|\gamma|}\dot\gamma\cdot\omega \right\} ds \\
&= 4\int_0^1 \left\{ h_\perp \cdot \dot h_\perp - h_\perp G \dot h_\perp \right\} \frac{|\dot\gamma|}{|\gamma|} ds
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\int_0^1 \dot\gamma (\nabla^2 G; h, h)\dot\gamma ds \\
&= \int_0^1 \dot\gamma \left\{ |\frac{\nabla P_\perp \cdot h}{|\gamma|} h\rangle\langle\omega| + |\omega\rangle\langle h \frac{\nabla P_\perp \cdot h}{|\gamma|}| + 2\nabla P_\perp \cdot h G \nabla P_\perp \cdot h \right\} \dot\gamma ds \\
&= 2\int_0^1 \left\{ -\frac{(\dot\gamma\cdot\omega)^2}{|\gamma|^2}|h_\perp|^2 + \frac{(\dot\gamma\cdot\omega)^2}{|\gamma|^2} h_\perp G h_\perp \right\} ds \\
&= 2\int_0^1 \left\{ -|h_\perp|^2 + h_\perp G h_\perp \right\} \frac{|\dot\gamma|^2}{|\gamma|^2} ds.
\end{aligned} \tag{4.3}$$

Thus, using for the first and last equations below that

$$4h_\perp \cdot \dot h_\perp |\dot\gamma|/|\gamma| - 2|h_\perp|^2|\dot\gamma|^2/|\gamma|^2 = 2(h_\perp^2|\dot\gamma|/|\gamma|)^{\cdot}, \tag{4.4}$$

we find

$$
\begin{aligned}
&\langle \partial_\kappa^2 E(x,0)h,h\rangle \\
&\quad = 2\int_0^1 \Big\{\dot h G\dot h - 2h_\perp G\dot h_\perp \frac{|\dot\gamma|}{|\gamma|} + h_\perp G h_\perp \frac{|\dot\gamma|^2}{|\gamma|^2}\Big\} ds \text{ (integrating by parts)}\\
&\quad = 2\int_0^1 \big\{|P\dot h|^2 + s^2\big(s^{-1}h_\perp\big)^{\cdot} G\big(s^{-1}h_\perp\big)^{\cdot}\big\} ds \text{ (using } |\dot\gamma|/|\gamma| = s^{-1})\\
&\quad \geq 2\int_0^1 \big\{|P\dot h|^2 + as^2\big|\big(s^{-1}h_\perp\big)^{\cdot}\big|^2\big\} ds \text{ (by (1.3))}\\
&\quad = 2\int_0^1 (|P\dot h|^2 + a|\big(h_\perp\big)^{\cdot}|^2) ds \text{ (integrating by parts).}
\end{aligned}
\tag{4.5}
$$

We conclude (using the bound $a \leq 1$) that

$$\langle \partial_\kappa^2 E(x,0)h,h\rangle \geq c\|h\|^2; \quad c \leq 2a \text{ and } h \in \mathcal{H}. \tag{4.6}$$

Hence G satisfies Definition 1.1 2) and therefore $\mathcal{O}_1 \subseteq \mathcal{U}$. □

We will extend the above proof to the case of geodesics $\gamma \in \mathcal{E}_x$ for metrics "near $\mathcal{O}$" a priori not knowing that geodesics are unique. To do this we need dynamical control of the geodesics and this will be provided under the additional condition (1.13). First we discuss the case of $G \in \mathcal{O}_1$ (as in Lemma 4.1). Let γ denote any non-constant (maximal) geodesic for such a metric (note that we are here not assuming $\gamma(0) = 0$ but only the differential equation (2.12)). Introduce the observables

$$A = \frac{\dot\gamma}{|\dot\gamma|_G} \cdot \hat\gamma \text{ and } B = \hat{\dot\gamma} \cdot \hat\gamma; \tag{4.7}$$

$$|\dot\gamma|_G = \sqrt{\dot\gamma G(\gamma)\dot\gamma},\ \ \hat\gamma = \frac{\gamma}{|\gamma|},\ \hat{\dot\gamma} = \frac{\dot\gamma}{|\dot\gamma|}.$$

Note that

$$A = q\cdot\hat\gamma;\ q = \frac{G^{1/2}(\gamma)\dot\gamma}{|\dot\gamma|_G}, \tag{4.8}$$

cf. (1.12). In particular

$$A^2, B^2 \leq 1. \tag{4.9}$$

Lemma 4.2. *Suppose $G \in \mathcal{O}_1$ and that γ is a corresponding non-constant geodesic. Then*

$$\dot A = \frac{|\dot\gamma|^2}{|\gamma|\,|\dot\gamma|_G}\langle P_\perp(\hat\gamma)\hat{\dot\gamma}, TP_\perp(\hat\gamma)\hat{\dot\gamma}\rangle;\ T = G(\gamma) + 2^{-1}\gamma\cdot\nabla G(\gamma). \tag{4.10}$$

In particular if $G \in \mathcal{O}$

$$\dot A \geq \bar c \frac{|\dot\gamma|_G}{|\gamma|}\Big(1 - A^2\Big), \tag{4.11}$$

with $\bar c > 0$ given by (1.13).

Proof. Using (2.12), (4.1a) and (4.1b) we compute (4.10). Note that the denominator $|\dot\gamma|_G$ is preserved, cf. (2.13).

As for (4.11) we use (4.10), (1.13) and

$$|P(\hat\gamma)\hat{\dot\gamma}|^2 = 1 - |P_\perp(\hat\gamma)\hat{\dot\gamma}|^2 = B^2 = \frac{|\dot\gamma|_G^2}{|\dot\gamma|^2}A^2. \tag{4.12}$$

□

Lemma 4.3. *Let $G \in \mathcal{O}_1$, and let A and B be given by (4.7) for any $(\gamma, \dot\gamma) \in (\mathbb{R}^d \setminus \{0\})^2$. We have*

$$b(1 - B^2) \geq 1 - A^2 \tag{4.13a}$$

$$a^{-1}(1 - A^2) \geq 1 - B^2. \tag{4.13b}$$

Proof. Using (1.12) and (4.12) we can estimate

$$\frac{|\dot\gamma|_G^2}{|\dot\gamma|^2} = |P\hat{\dot\gamma}|^2 + |P_\perp G^{1/2} P_\perp \hat{\dot\gamma}|^2 \leq 1 + (b-1)(1 - B^2), \tag{4.14}$$

which in turn using (4.9), (4.12) and the fact that $b \geq 1$ yields

$$1 - B^2 \geq 1 - A^2 - (b-1)(1 - B^2). \tag{4.15}$$

Obviously (4.13a) follows from (4.15).

Next we mimic the proof of (4.13a). We have $1 - A^2 = 1 - \frac{|\dot\gamma|^2}{|\dot\gamma|_G^2} B^2$, and letting $q = G^{1/2}\dot\gamma / |\dot\gamma|_G$ (as in (4.8)) we estimate

$$\frac{|\dot\gamma|^2}{|\dot\gamma|_G^2} = |Pq|^2 + |P_\perp G^{-1/2} q|^2 \leq 1 + (a^{-1} - 1)(1 - A^2),$$

cf. (4.14). Whence the analogue of (4.15) holds, and we conclude (4.13b). □

We remark that only (4.13b) will be needed. The estimate (4.13a) is given only for completeness of presentation.

4.2. **Perturbed case.** Now let $G \in \mathcal{O}$ be given. We shall use the notation G_ϵ to denote any metric of order zero obeing $\|G_\epsilon - G\|_l \leq \epsilon$. The positive parameter ϵ is an order parameter which we will take sufficiently small, say $\epsilon \leq \epsilon_0$, in terms of quantities given by the fixed unperturbed G. We shall use the observables A and B of (4.7) defined in terms of G but now evaluated at $\gamma \to \gamma_\epsilon$; clearly they are well-defined for $\gamma_\epsilon(s), \dot\gamma_\epsilon(s) \neq 0$.

Lemma 4.4. *There exist $\epsilon_0 > 0$ and $C_1, C_2, C_3 > 0$ such that if $\|G_\epsilon - G\|_l \leq \epsilon \leq \epsilon_0$ and γ_ϵ is any non-constant geodesic for the metric G_ϵ with $\gamma_\epsilon(0) = 0$, then*

i) $\gamma_\epsilon(s) \neq 0$ for all $s > 0$.

ii) $B = B(\gamma_\epsilon(s), \dot\gamma_\epsilon(s)) \geq 1 - \epsilon C_1$ for all $s > 0$.

iii) There are bounds

$$(1 - \epsilon C_2)|\dot\gamma_\epsilon(0)|^2 \leq |\dot\gamma_\epsilon(s)|^2 \leq (1 + \epsilon C_2)|\dot\gamma_\epsilon(0)|^2, \tag{4.16a}$$

$$(1 - \epsilon C_3)|s\dot\gamma_\epsilon(0)|^2 \leq |\gamma_\epsilon(s)|^2 \leq (1 + \epsilon C_3)|s\dot\gamma_\epsilon(0)|^2. \tag{4.16b}$$

Proof. **Re i).** Let γ_ϵ be any such geodesic. Being non-constant implies $\dot\gamma_\epsilon(0) \neq 0$. Indeed note at this point that the quantity $\dot\gamma_\epsilon G_\epsilon(\gamma_\epsilon)\dot\gamma_\epsilon$ is constant. Moreover, using also (1.3) and the fact that $G(0) = I$,

$$a|\dot\gamma_\epsilon(s)|^2 \leq |\dot\gamma_\epsilon(s)|^2_{G(\gamma_\epsilon(s))} \leq b|\dot\gamma_\epsilon(s)|^2, \tag{4.17a}$$

$$(1 - \epsilon C)|\dot\gamma_\epsilon(0)|^2 \leq |\dot\gamma_\epsilon(s)|^2_{G(\gamma_\epsilon(s))} \leq (1 + \epsilon C)|\dot\gamma_\epsilon(0)|^2, \tag{4.17b}$$

$$(1 - \epsilon C)b^{-1}\dot\gamma_\epsilon(0)|^2 \leq |\dot\gamma_\epsilon(s)|^2 \leq (1 + \epsilon C)a^{-1}|\dot\gamma_\epsilon(0)|^2. \tag{4.17c}$$

We shall use (4.17a) and (4.17b) later in the proof while (4.17c) is stated just for completeness (note that (4.16a) is stronger than (4.17c)).

Whence the observables A and B are well-defined at this geodesic on an interval of the form $]0, s_0[$. We need to show that s_0 can be taken arbitrarily large. Suppose not, then we let s_0 be the first positive nullpoint, $\gamma_\epsilon(s_0) = 0$, and we need to find a contradiction. For that it suffices show the bound

$$A(s) \geq 1 - \epsilon\bar{C} \text{ for all } s \in]0, s_0[, \tag{4.18}$$

where the constant $\bar{C} > 0$ depends only on G (i.e. it is independent of ϵ, G_ϵ and γ_ϵ). In particular we can assume that $\epsilon\bar{C} < 1$. It then follows that $A, B > 0$. Whence by the computation

$$\tfrac{d}{ds}|\gamma_\epsilon|^2 = 2\langle\gamma_\epsilon, \dot{\gamma}_\epsilon\rangle = 2|\gamma_\epsilon|\,|\dot{\gamma}_\epsilon|B > 0, \tag{4.19}$$

we see that $s \to |\gamma_\epsilon(s)|$ is increasing yielding the contradiction and whence showing i).

It remains to show (4.18). We compute the time-derivative of A and find the following extension of (4.11).

$$\dot{A} \geq \bar{c}\frac{|\dot{\gamma}_\epsilon|_{G(\gamma_\epsilon(s))}}{|\gamma_\epsilon|}\big(1 - A^2\big) - \epsilon\tilde{C}\frac{|\dot{\gamma}_\epsilon|}{|\gamma_\epsilon|}. \tag{4.20}$$

Note that

$$\tfrac{d}{ds}|\dot{\gamma}_\epsilon|_G^{-1} = \frac{\tfrac{d}{ds}\{\dot{\gamma}_\epsilon\big(G_\epsilon - G\big)\dot{\gamma}_\epsilon\}}{2|\dot{\gamma}_\epsilon|_G^3},$$

and whence due to (4.17a) that

$$|\tfrac{d}{ds}|\dot{\gamma}_\epsilon|_G^{-1}| \leq \epsilon\breve{C}|\gamma_\epsilon|^{-1}.$$

The constant $\breve{C}$ contributes to the constant $\tilde{C}$ in (4.20). Another term of the form of the last term in (4.20) comes from comparing the right hand side of the geodesic equation (2.12) for γ_ϵ with the same expression replacing $G_\epsilon(\gamma_\epsilon) \to G(\gamma_\epsilon)$.

Using (4.17a) again we can simplify (4.20) as

$$\dot{A} \geq \bar{c}\sqrt{a}\frac{|\dot{\gamma}_\epsilon|}{|\gamma_\epsilon|}\big(1 - A^2 - \epsilon K\big);\ K = \tfrac{\tilde{C}}{\bar{c}\sqrt{a}}. \tag{4.21}$$

Combining (4.21) and the fact that $\lim_{s\to 0^+} A(s) = 1$ (here once more using that $G(0) = I$), we obtain (4.18) for any $\bar{C} > K/2$ provided ϵ is chosen small enough (precisely we need $2\bar{C} - K > \epsilon\bar{C}^2$).

Re ii). We write $B = 1 - (1 - B)$ and then use (4.13b) and (4.18) (we use (4.18) with $s_0 = \infty$) to estimate

$$(1 + B)B \geq (1 + B) - \epsilon\bar{C}a^{-1}(1 + A) \geq 1 + B - \epsilon 2\bar{C}a^{-1}. \tag{4.22}$$

We subtract B from (4.22) and obtain

$$B \geq \sqrt{1 - \epsilon 2\bar{C}a^{-1}} \geq 1 - \epsilon\bar{C}a^{-1} - \epsilon^2 C \geq 1 - \epsilon C_1;\ C_1 := 2\bar{C}a^{-1}.$$

Re iii). Using ii) we obtain, cf. (4.14) and (4.22),

$$\big(1 - \epsilon 2C_1\big)|\dot{\gamma}_\epsilon(s)|^2 \leq |\dot{\gamma}_\epsilon(s)|^2_{G(\gamma_\epsilon(s))}.$$

Also, using (4.18), we have

$$(1 - \epsilon\bar{C})|\dot{\gamma}_\epsilon(s)|_{G(\gamma_\epsilon(s))} \leq |\dot{\gamma}_\epsilon(s)|_{G(\gamma_\epsilon(s))}A \leq |\dot{\gamma}_\epsilon(s)|,$$

yielding

$$|\dot{\gamma}_\epsilon(s)|^2_{G(\gamma_\epsilon(s))} \leq (1 - \epsilon\bar{C})^{-2}|\dot{\gamma}_\epsilon(s)|^2.$$

Using the shown two-sided estimates in combination with (4.17b) we obtain (4.16a).
Using

$$\tfrac{d}{ds}|\gamma_\epsilon| = |\dot\gamma_\epsilon|B,$$

cf. (4.19), we obtain (4.16b) from (4.16a) and ii) by integration. □

We have the following version of Definition 1.1 2) for the perturbed metrics.

Lemma 4.5. *There exists $\epsilon_0 > 0$ (possibly smaller than the ϵ_0 of Lemma 4.4) such that if $\|G_\epsilon - G\|_l \le \epsilon \le \epsilon_0$, $x \in \mathbb{R}^d$ and γ_ϵ is any geodesic for the metric G_ϵ emanating from 0 with value x at time one, then writing $\gamma_\epsilon(s) = sx + \kappa(s)$, denoting by E_ϵ the energy functional (1.4) with G replaced by G_ϵ and using the positive number a given by (1.3) for the metric G,*

$$\langle \partial_\kappa^2 E_\epsilon(x,\kappa)h, h\rangle \ge a\|h\|^2, \qquad h \in \mathcal{H}. \tag{4.23}$$

Proof. Note that for $x = 0$ the geodesic γ_ϵ is unique due to Lemma 4.4 i), it is given by $\gamma_\epsilon = 0$. By (1.8) $\partial_\kappa^2 E_\epsilon(0,0) = 2G_\epsilon(0)$, so obviously (4.23) holds in this case provided $a \le 2a - 2\epsilon d$ $(\le 2G_\epsilon(0))$.

Suppose next that $x \neq 0$. We can mimic the proof of Lemma 4.1 using Lemma 4.4. By letting $s = 1$ in (4.16b) we obtain from (4.16a) and (4.16b) that there exist $C_4, C_5 > 0$ such that

$$(1-\epsilon C_4)|x|^2 \le |\dot\gamma_\epsilon(s)|^2 \le (1+\epsilon C_4)|x|^2, \tag{4.24a}$$

$$(1-\epsilon C_5)|sx|^2 \le |\gamma_\epsilon(s)|^2 \le (1+\epsilon C_5)|sx|^2. \tag{4.24b}$$

A consequence of (4.24a) and (4.24b) is that the quantity $|\dot\gamma_\epsilon|/|\gamma_\epsilon|$ appearing when we try to repeat the proof of Lemma 4.1 effectively is given by s^{-1}, to be used in the last part of the proof only. More precisely

$$(1-\epsilon C_6)s^{-1} \le |\dot\gamma_\epsilon|/|\gamma_\epsilon| \le (1+\epsilon C_6)s^{-1}. \tag{4.25}$$

We calculate using (4.1b), (4.12), Lemma 4.4 ii) and the notation $\omega = \gamma_\epsilon/|\gamma_\epsilon|$

$$P_\perp \dot\gamma_\epsilon = O(\sqrt{\epsilon})\,|\dot\gamma_\epsilon|, \tag{4.26a}$$

$$\tfrac{d}{ds}P(\gamma_\epsilon) = \left|\frac{P_\perp\dot\gamma_\epsilon}{|\gamma_\epsilon|}\right\rangle\langle\omega| + |\omega\rangle\left\langle\frac{P_\perp\dot\gamma_\epsilon}{|\gamma_\epsilon|}\right| = O(\sqrt{\epsilon})\,\tfrac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|}; \tag{4.26b}$$

here and below the notation $O(\sqrt{\epsilon})$ is used for any (matrix-valued) function of $s \in [0,1]$ obeying $|O(\sqrt{\epsilon})_{ij}| \le \sqrt{\epsilon}C$ uniformly in s, x, G_ϵ and γ_ϵ. In fact we can above choose $C = \sqrt{2C_1}$ and $C = 2\sqrt{2C_1}$, respectively.

From (4.26b) we obtain

$$\tfrac{d}{ds}\big(P(\gamma_\epsilon)h\big) = P(\gamma_\epsilon)\dot h + O(\sqrt{\epsilon})\,\tfrac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|}h. \tag{4.27}$$

Below we stick to the notation $\dot h_\perp = (h_\perp)^{\cdot}$, although by (4.27) we could have choosen an alternative interpretation.

Now, using (4.1a), (4.1b), (4.26a), (4.27) and Lemma 4.4 ii) the analogue of (4.2) reads

$$4\int_0^1 \dot\gamma_\epsilon \nabla G_\epsilon \cdot h\dot h ds$$

$$= 4\int_0^1 \big\{h_\perp\cdot\dot h_\perp - h_\perp G\dot h_\perp + hO(\sqrt{\epsilon})\dot h + hO(\sqrt{\epsilon})h\tfrac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|}\big\}\frac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|}ds. \tag{4.28}$$

Similarly the analogue of (4.3) reads

$$\int_0^1 \dot\gamma_\epsilon(\nabla^2 G_\epsilon; h,h)\dot\gamma_\epsilon ds = 2\int_0^1 \big\{ -|h_\perp|^2 + h_\perp G h_\perp + hO(\sqrt{\epsilon})h\big\}\frac{|\dot\gamma_\epsilon|^2}{|\gamma_\epsilon|^2}ds. \tag{4.29}$$

Next we compute using (2.12), (4.1a), (4.1b) and Lemma 4.4 ii)

$$(|\dot\gamma_\epsilon|/|\gamma_\epsilon|)^\cdot = -(|\dot\gamma_\epsilon|/|\gamma_\epsilon|)^2\big(1+O(\sqrt{\epsilon})\big).$$

Whence the analogue of (4.4) reads

$$4h_\perp\cdot\dot h_\perp|\dot\gamma_\epsilon|/|\gamma_\epsilon| - 2|h_\perp|^2(|\dot\gamma_\epsilon|/|\gamma_\epsilon|)^2 = 2(h_\perp^2|\dot\gamma_\epsilon|/|\gamma_\epsilon|)^\cdot + hO(\sqrt{\epsilon})h(|\dot\gamma_\epsilon|/|\gamma_\epsilon|)^2. \tag{4.30}$$

We insert (4.28) and (4.29) into (1.7), integrate by parts using (4.30) and obtain the following partial analogue of (4.5)

$$\begin{aligned}&\langle\partial_\kappa^2 E_\epsilon(x,\kappa)h,h\rangle \qquad (4.31)\\ &= 2\int_0^1\Big\{\dot h G_\epsilon \dot h - 2h_\perp G\dot h_\perp\frac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|} + h_\perp G h_\perp\frac{|\dot\gamma_\epsilon|^2}{|\gamma_\epsilon|^2} + hO(\sqrt{\epsilon})\dot h\frac{|\dot\gamma_\epsilon|}{|\gamma_\epsilon|} + hO(\sqrt{\epsilon})h\frac{|\dot\gamma_\epsilon|^2}{|\gamma_\epsilon|^2}\Big\}ds.\end{aligned}$$

Finally we invoke (4.25) and (3.11a), and obtain from (4.31) using again (4.27) the following analogue of (4.6)

$$\langle\partial_\kappa^2 E_\epsilon(x,\kappa)h,h\rangle \geq 2\int_0^1 (a-\sqrt{\epsilon}C)|\dot h|^2 \geq a\|h\|^2. \tag{4.32}$$

□

Proof of Theorem 1.4 i). Let $\Phi_\epsilon = \exp_{0,\epsilon}(1\cdot) : T\mathbb{R}^d_0 \to \mathbb{R}^d$ denote the exponential map for the perturbed metric G_ϵ (close to a given $G\in\mathcal{O}$) at the point $0\in\mathbb{R}^d$ and evaluated at time one. The positivity of the Hessian along any perturbed geodesic as guaranteed by Lemma 4.5 implies that Φ_ϵ is a local diffeomorphism, cf. [Ch, Theorem 2.16] and [Mi, Theorem 14.1]. By Lemma 2.1 Φ_ϵ maps $T\mathbb{R}^d_0$ onto $\mathbb{R}^d$. By (4.16b) Φ_ϵ is proper, whence Φ_ϵ is one-to-one, cf. [Be, Theorem 5.1.4]. We have verified the uniqueness property of Definition 1.1 1).

The uniform positivity property of Definition 1.1 2) follows from Lemma 4.5 (with $c=a$ in (1.7)). Whence indeed $G_\epsilon\in\mathcal{U}$ for $\epsilon\le\epsilon_0$. □

5. Proof of Theorem 1.4 ii)

We shall prove the bounds (1.15a), (1.16) and (1.15c). Let $G\in\mathcal{O}$ be given. Using the convention of Subsection 4.2 we write $\tilde G = G_\epsilon$ for perturbed metrics, and we shall again require $\|G_\epsilon - G\|_l \le \epsilon\le\epsilon_0$ for some small $\epsilon_0>0$.

Re (1.15a). By the variational definition (1.5) and (2.5), $S^2(x)-|x|^2 = O(\epsilon)|x|^2$ uniformly in $x\in\mathbb{R}^d\setminus\{0\}$. Whence we obtain uniformly in $x\in\mathbb{R}^d\setminus\{0\}$

$$S(x)-|x| = (S^2(x)-|x|^2)/(S(x)+|x|) = O(\epsilon)|x|, \tag{5.1}$$

yielding (1.15a) by division with $|x|$.

Re (1.16). We compute

$$\partial_i s(x) = \frac{\partial_i S(x)}{|x|} - \frac{x_i S(x)}{|x|^3}. \tag{5.2}$$

For the first term we use (3.5), (1.15a), (4.24a) and Lemma 4.4 ii) to write

$$|x|\partial_i S(x) = x_i + O(\sqrt{\epsilon})|x|. \tag{5.3}$$

As for second term we use (1.15a) to write

$$\frac{x_i S(x)}{|x|} = x_i + O(\epsilon)|x|. \tag{5.4}$$

Clearly (1.16) follows from (5.2)–(5.4).

Re (1.15c).

Step I. We prove the uniform bound

$$\|\partial_x^\alpha \kappa\| \leq \sqrt{\epsilon} C \text{ for } |\alpha| = 1. \tag{5.5}$$

Note that (5.5) without the factor $\sqrt{\epsilon}$ to the right follows from Proposition 3.1. Due to Lemma 4.5 it suffices to bound the expression (3.8a) for $n = 1$ as

$$|\langle \partial_x^\alpha \partial_\kappa E_\epsilon(x, \kappa), h\rangle| \leq \sqrt{\epsilon} C \|h\| \text{ for } \alpha = e_j. \tag{5.6}$$

As in the proof of Proposition 3.1 we compute the x-derivative using (2.7) yielding four terms. Up to an error of order $O(\epsilon)\|h\|$ we can replace $G_\epsilon \to G$, and whence it suffices to show that the sum of the following four expressions is of order $O(\sqrt{\epsilon})\|h\|$:

$$\begin{aligned}
T_1 &= \int_0^1 2e_j \nabla G \cdot h\dot{\gamma}_\epsilon \, ds, \\
T_2 &= \int_0^1 2\dot{h} G e_j \, ds, \\
T_3 &= \int_0^1 s\dot{\gamma}_\epsilon \nabla \partial_j G \cdot h\dot{\gamma}_\epsilon \, ds, \\
T_4 &= \int_0^1 2s\dot{h} \partial_j G \dot{\gamma}_\epsilon \, ds.
\end{aligned}$$

To do this we use (3.11a), (4.1a), (4.1b), (4.12), (4.25) and Lemma 4.4 ii) and obtain

$$\begin{aligned}
T_1 &= \int_0^1 2e_j (P_\perp - P_\perp G P_\perp) \tfrac{h}{s} \, ds + O(\sqrt{\epsilon})\|h\|, \\
T_2 &= \int_0^1 2e_j (P + P_\perp G P_\perp) \dot{h} \, ds, \\
T_3 &= \int_0^1 2e_j (P_\perp G P_\perp - P_\perp) \tfrac{h}{s} \, ds + O(\sqrt{\epsilon})\|h\|, \\
T_4 &= \int_0^1 2e_j (P_\perp - P_\perp G P_\perp) \dot{h} \, ds + O(\sqrt{\epsilon})\|h\|.
\end{aligned}$$

Clearly it follows that

$$\begin{aligned}
T_1 + T_3 &= O(\sqrt{\epsilon})\|h\|, \\
T_2 + T_4 &= \int_0^1 2e_j \dot{h} \, ds + O(\sqrt{\epsilon})\|h\| = O(\sqrt{\epsilon})\|h\|.
\end{aligned}$$

Whence we have proved (5.6).

Step II. We shall prove the uniform bound

$$|\partial_x^\alpha \dot{\kappa}(1)| = |\partial_x^\alpha \dot{\gamma}_\epsilon(1) - e_j| \leq \sqrt{\epsilon} C \text{ for } \alpha = e_j. \tag{5.7}$$

We claim that

$$\partial_j \ddot{\kappa}(s) = s^{-1}F(s) \text{ where } \int_0^1 |F(s)|^2\, ds \le \epsilon C^2. \tag{5.8}$$

Note that (5.7) follows from (3.7), (5.5) and (5.8). The difficulty here is not to show that the quantity $s\partial_j\ddot{\kappa}(s)$ is square integrable but rather to show that its L^2-norm is bounded by $\sqrt{\epsilon}C$ as stated in (5.8). We shall show that

$$\partial_j\ddot{\kappa} = G^{-1}P_\perp \nabla G\cdot\dot{\gamma}_\epsilon P_\perp\big(s^{-1}\partial_j\kappa - \partial_j\dot{\kappa}\big) + s^{-1}\tilde{F} \text{ where } \|\tilde{F}\|_{L^2} \le \sqrt{\epsilon}C, \tag{5.9}$$

which combined with (4.25) and (5.5) supplies (5.8).

To prove (5.9) we proceed as follows. Differentiating (2.12) with respect to x_j and using (4.25), (3.11a), (5.5) and the facts that $\partial_j\gamma_\epsilon = se_j + \partial_j\kappa$ and $\|G_\epsilon - G\|_l \le \epsilon$ we obtain

$$\begin{aligned} G_\epsilon\partial_j\ddot{\kappa} = {} & \partial_j\dot{\gamma}_\epsilon\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon + 2^{-1}\dot{\gamma}_\epsilon\partial_j\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon - \partial_j\nabla G\cdot\dot{\gamma}_\epsilon\dot{\gamma}_\epsilon \\ & - \nabla G\cdot\partial_j\dot{\gamma}_\epsilon\dot{\gamma}_\epsilon - \nabla G\cdot\dot{\gamma}_\epsilon\partial_j\dot{\gamma}_\epsilon - \partial_j G\ddot{\kappa} + O(\epsilon)/s, \end{aligned} \tag{5.10}$$

where here and henceforth $O(\epsilon^p)$ stands for a function with L^2-norm bounded by $\epsilon^p C$.

In the remaining of the proof of (5.9) we will repeatedly use (4.1a), (4.1b), (4.25), (4.26a), (3.11a) and (5.5).

First we estimate $\partial_j G\ddot{\kappa}$. To this end we note as above that

$$\partial_j G\ddot{\kappa} = \partial_j GG^{-1}(2^{-1}\dot{\gamma}_\epsilon\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon - \nabla G\cdot\dot{\gamma}_\epsilon\dot{\gamma}_\epsilon) + O(\epsilon)/s.$$

We compute

$$\begin{aligned} \partial_j GG^{-1}\dot{\gamma}_\epsilon\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon &= \nabla G\cdot\partial_j\gamma_\epsilon G^{-1}[\dot{\gamma}_\epsilon\nabla P\cdot e_\bullet\dot{\gamma}_\epsilon + \dot{\gamma}_\epsilon P_\perp\nabla G\cdot e_\bullet P_\perp\dot{\gamma}_\epsilon \\ &\qquad + \dot{\gamma}_\epsilon\nabla P_\perp\cdot e_\bullet GP_\perp\dot{\gamma}_\epsilon + \dot{\gamma}_\epsilon P_\perp G\nabla P_\perp\cdot e_\bullet\dot{\gamma}_\epsilon] \\ &= \nabla G\cdot\partial_j\gamma_\epsilon G^{-1}\dot{\gamma}_\epsilon\nabla P\cdot e_\bullet\dot{\gamma}_\epsilon + O(\sqrt{\epsilon})/s \\ &= O(\sqrt{\epsilon})/s. \end{aligned}$$

Similarly we obtain $\partial_j GG^{-1}\nabla G\cdot\dot{\gamma}_\epsilon\dot{\gamma}_\epsilon = O(\sqrt{\epsilon})/s$, and therefore

$$\partial_j G\ddot{\kappa} = O(\sqrt{\epsilon})/s. \tag{5.11}$$

Now we estimate the first five terms on the right side of (5.10).

(i) For the first term we have

$$\begin{aligned} \partial_j\dot{\gamma}_\epsilon\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon &= \partial_j\dot{\gamma}_\epsilon[\nabla P\cdot e_\bullet + P_\perp\nabla G\cdot e_\bullet P_\perp + \nabla P_\perp\cdot e_\bullet GP_\perp + P_\perp G\nabla P_\perp\cdot e_\bullet]\dot{\gamma}_\epsilon \\ &= \partial_j\dot{\gamma}_\epsilon\nabla P\cdot e_\bullet\dot{\gamma}_\epsilon + \partial_j\dot{\gamma}_\epsilon P_\perp G\nabla P_\perp\cdot e_\bullet\dot{\gamma}_\epsilon + O(\sqrt{\epsilon})/s \\ &= (\langle\hat{\gamma}_\epsilon,\dot{\gamma}_\epsilon\rangle/|\gamma_\epsilon|)(\langle\partial_j\dot{\gamma}_\epsilon, P_\perp e_\bullet\rangle - \langle GP_\perp\partial_j\dot{\gamma}_\epsilon, P_\perp e_\bullet\rangle) + O(\sqrt{\epsilon})/s. \end{aligned}$$

Thus,

$$\partial_j\dot{\gamma}_\epsilon\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon = (\langle\hat{\gamma}_\epsilon,\dot{\gamma}_\epsilon\rangle/|\gamma_\epsilon|)(I - P_\perp G)P_\perp\partial_j\dot{\gamma}_\epsilon + O(\sqrt{\epsilon})/s. \tag{5.12}$$

(ii) To estimate the second term we consider

$$\begin{aligned} \dot{\gamma}_\epsilon\partial_j\nabla G\cdot e_\bullet\dot{\gamma}_\epsilon &= \dot{\gamma}_\epsilon\nabla[\nabla P\cdot e_\bullet + P_\perp\nabla G\cdot e_\bullet P_\perp \\ &\qquad + \nabla P_\perp\cdot e_\bullet GP_\perp + P_\perp G\nabla P_\perp\cdot e_\bullet]\cdot\partial_j\gamma_\epsilon\dot{\gamma}_\epsilon \\ &= \dot{\gamma}_\epsilon\nabla(\nabla P\cdot e_\bullet)\cdot\partial_j\gamma_\epsilon\dot{\gamma}_\epsilon + \dot{\gamma}_\epsilon\nabla(\nabla P_\perp\nabla\cdot e_\bullet GP_\perp)\cdot\partial_j\gamma_\epsilon\dot{\gamma}_\epsilon \\ &\qquad + \dot{\gamma}_\epsilon\nabla(P_\perp G\nabla P_\perp\cdot e_\bullet)\cdot\partial_j\gamma_\epsilon\dot{\gamma}_\epsilon + O(\sqrt{\epsilon})/s. \end{aligned}$$

It follows that

$$\dot\gamma_\epsilon \partial_j \nabla G \cdot e_\bullet \dot\gamma_\epsilon = \dot\gamma_\epsilon \nabla(\nabla P \cdot e_\bullet) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon + \dot\gamma_\epsilon \nabla P \cdot e_\bullet G \nabla P \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon \\ + \dot\gamma_\epsilon \nabla P \cdot \partial_j \gamma_\epsilon G \nabla P \cdot e_\bullet \dot\gamma_\epsilon + O(\sqrt{\epsilon})/s. \tag{5.13}$$

Using the fact that for any fixed vectors h and z we have $\nabla(\hat{x}) \cdot h = P_\perp h/|\gamma_\epsilon|$ and

$$\nabla \left(\frac{P_\perp z}{|x|} \right) \cdot h = -\frac{1}{|x|^3} \left[\langle x, z \rangle P_\perp h + \langle P_\perp h, z \rangle x + \langle x, h \rangle P_\perp z \right],$$

we find that

$$\begin{aligned} \dot\gamma_\epsilon \nabla(\nabla P \cdot e_\bullet) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon &= \dot\gamma_\epsilon \nabla \left[\Big| \frac{P_\perp e_\bullet}{|\gamma_\epsilon|} \Big\rangle \langle \hat\gamma_\epsilon | + |\hat\gamma_\epsilon \rangle \Big\langle \frac{P_\perp e_\bullet}{|\gamma_\epsilon|} \Big| \right] \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon \\ &= 2 \langle \hat\gamma_\epsilon, \dot\gamma_\epsilon \rangle \left\langle \dot\gamma_\epsilon, \nabla \left(\frac{P_\perp e_\bullet}{|\gamma_\epsilon|} \right) \cdot \partial_j \gamma_\epsilon \right\rangle + O(\sqrt{\epsilon})/s \\ &= -2 \langle \hat\gamma_\epsilon, \dot\gamma_\epsilon \rangle \left\langle \dot\gamma_\epsilon, \frac{\langle P_\perp \partial_j \gamma_\epsilon, e_\bullet \rangle \gamma_\epsilon}{|\gamma_\epsilon|^3} \right\rangle + O(\sqrt{\epsilon})/s. \end{aligned}$$

Hence

$$\dot\gamma_\epsilon \nabla(\nabla P \cdot e_\bullet) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon = -2 \frac{\langle \dot\gamma_\epsilon, \hat\gamma_\epsilon \rangle^2}{|\gamma_\epsilon|^2} P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s.$$

Moreover, using twice (4.1b) we obtain

$$\begin{aligned} \dot\gamma_\epsilon \nabla P \cdot \partial_j \gamma_\epsilon G \nabla P \cdot e_\bullet \dot\gamma_\epsilon &= \dot\gamma_\epsilon \left[\Big| \frac{P_\perp \partial_j \gamma_\epsilon}{|\gamma_\epsilon|} \Big\rangle \langle \hat\gamma_\epsilon | + |\hat\gamma_\epsilon \rangle \Big\langle \frac{P_\perp \partial_j \gamma_\epsilon}{|\gamma_\epsilon|} \Big| \right] G \nabla P \cdot e_\bullet \dot\gamma_\epsilon \\ &= \langle \dot\gamma_\epsilon, \hat\gamma_\epsilon \rangle \left\langle \frac{G P_\perp \partial_j \gamma_\epsilon}{|\gamma_\epsilon|}, \nabla P \cdot e_\bullet \dot\gamma_\epsilon \right\rangle + O(\sqrt{\epsilon})/s \\ &= (\langle \dot\gamma_\epsilon, \hat\gamma_\epsilon \rangle^2 / |\gamma_\epsilon|^2) P_\perp G P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s. \end{aligned}$$

A similar calculation supplies

$$\dot\gamma_\epsilon \nabla P \cdot e_\bullet G \nabla P \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon = (\langle \dot\gamma_\epsilon, \hat\gamma_\epsilon \rangle^2 / |\gamma_\epsilon|^2) P_\perp G P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s,$$

and putting the last three estimates together in (5.13) gives

$$2^{-1} \dot\gamma_\epsilon \partial_j \nabla G \cdot e_\bullet \dot\gamma_\epsilon = (\langle \dot\gamma_\epsilon, \hat\gamma_\epsilon \rangle^2 / |\gamma_\epsilon|^2)(P_\perp G - I) P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s \tag{5.14}$$

(iii) Concerning the third term of (5.10) we have

$$\begin{aligned} \partial_j \nabla G \cdot \dot\gamma_\epsilon \dot\gamma_\epsilon &= \nabla(\nabla P \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon + \nabla(P_\perp \nabla G \cdot \dot\gamma_\epsilon P_\perp) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon \\ &\quad + \nabla(\nabla P_\perp \cdot \dot\gamma_\epsilon G P_\perp) \cdot \partial \gamma_\epsilon \dot\gamma_\epsilon + \nabla(P_\perp G \nabla P_\perp \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon \\ &= \nabla(\nabla P \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon + P_\perp \nabla G \cdot \dot\gamma_\epsilon \nabla P_\perp \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon \\ &\quad + P_\perp G \nabla(\nabla P_\perp \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon + O(\sqrt{\epsilon})/s. \end{aligned}$$

Analogous calculations to the ones leading to (5.14) yield

$$\begin{aligned} \nabla(\nabla P \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon &= -(\langle \hat\gamma_\epsilon, \dot\gamma_\epsilon \rangle / |\gamma_\epsilon|)^2 P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s, \\ P_\perp \nabla G \cdot \dot\gamma_\epsilon \nabla P_\perp \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon &= -(\langle \hat\gamma_\epsilon, \dot\gamma_\epsilon \rangle / |\gamma_\epsilon|) P_\perp \nabla G \cdot \dot\gamma_\epsilon P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s, \end{aligned}$$

and

$$P_\perp G \nabla(\nabla P_\perp \cdot \dot\gamma_\epsilon) \cdot \partial_j \gamma_\epsilon \dot\gamma_\epsilon = (\langle \hat\gamma_\epsilon, \dot\gamma_\epsilon \rangle / |\gamma_\epsilon|)^2 P_\perp G P_\perp \partial_j \gamma_\epsilon + O(\sqrt{\epsilon})/s.$$

Therefore

$$\begin{aligned}\partial_j \nabla G\cdot\dot\gamma_\epsilon\dot\gamma_\epsilon = &-(\langle\hat\gamma_\epsilon,\dot\gamma_\epsilon\rangle/|\gamma_\epsilon|)^2P_\perp\partial_j\gamma_\epsilon-(\langle\hat\gamma_\epsilon,\dot\gamma_\epsilon\rangle/|\gamma_\epsilon|)P_\perp\nabla G\cdot\dot\gamma_\epsilon P_\perp\partial_j\gamma_\epsilon\\ &+(\langle\hat\gamma_\epsilon,\dot\gamma_\epsilon\rangle/|\gamma_\epsilon|)^2P_\perp GP_\perp\partial_j\gamma_\epsilon+O(\sqrt\epsilon)/s,\end{aligned}\tag{5.15}$$

(iv) To estimate the fourth term of (5.10) we proceed as follows

$$\begin{aligned}\nabla G\cdot\partial_j\dot\gamma_\epsilon\dot\gamma_\epsilon &= (\nabla P\cdot\partial_j\dot\gamma_\epsilon+P_\perp\nabla G\cdot\partial_j\dot\gamma_\epsilon P_\perp\\ &\quad+\nabla P_\perp\cdot\partial_j\dot\gamma_\epsilon GP_\perp+P_\perp G\nabla P_\perp\cdot\partial_j\dot\gamma_\epsilon)\dot\gamma_\epsilon\\ &=\nabla P\cdot\partial_j\dot\gamma_\epsilon+P_\perp G\nabla P_\perp\cdot\partial_j\dot\gamma_\epsilon\dot\gamma_\epsilon+O(\sqrt\epsilon)/s\\ &=(\langle\hat\gamma_\epsilon,\dot\gamma_\epsilon\rangle/|\gamma_\epsilon|)(I-P_\perp G)P_\perp\partial_j\dot\gamma_\epsilon+O(\sqrt\epsilon)/s.\end{aligned}\tag{5.16}$$

(v) For the fifth term of (5.10) we have

$$\begin{aligned}\nabla G\cdot\dot\gamma_\epsilon\partial_j\dot\gamma_\epsilon &= (\nabla P\cdot\dot\gamma_\epsilon+P_\perp\nabla G\cdot\dot\gamma_\epsilon P_\perp\\ &\quad+\nabla P_\perp\cdot\dot\gamma_\epsilon GP_\perp+P_\perp G\nabla P_\perp\cdot\dot\gamma_\epsilon)\partial_j\dot\gamma_\epsilon\\ &=P_\perp\nabla G\cdot\dot\gamma_\epsilon P_\perp\partial_j\dot\gamma_\epsilon+O(\sqrt\epsilon)/s.\end{aligned}\tag{5.17}$$

Substituting (5.11), (5.12), (5.14), (5.15), (5.16) and (5.17) in (5.10) we find that

$$G_\epsilon\partial_j\breve\kappa=P_\perp\nabla G\cdot\dot\gamma_\epsilon P_\perp\left((\langle\hat\gamma_\epsilon,\dot\gamma_\epsilon\rangle/|\gamma_\epsilon|)\partial_j\gamma_\epsilon-\partial_j\dot\gamma_\epsilon\right)+O(\sqrt\epsilon)/s,$$

and using now (4.25) and Lemma 4.4 ii) we arrive at (5.9) due to cancellation.

Step III. We shall complete the proof of (1.15c) by differentiating (5.2) to obtain a representation of the second order derivatives and then using (1.15a), (1.16), (3.5) and (5.7). We have

$$\partial_{ij}s(x)=\frac{\partial_{ij}S}{|x|}-\frac{x_j\partial_iS+x_i\partial_jS}{|x|^3}-\frac{\delta_{ij}S}{|x|^3}+3\frac{x_ix_jS}{|x|^5}.\tag{5.18}$$

Using (1.15a) and (1.16) it follows from (5.18) that

$$\partial_{ij}s(x)-\frac{\partial_{ij}S}{|x|}=\frac{\hat x_i\hat x_j-\delta_{ij}+O(\sqrt\epsilon)}{|x|^2}.\tag{5.19}$$

As for the first term on the right hand side of (5.18) we use (3.5) and (5.7) to write

$$\frac{\partial_j\nabla S}{|x|}-\frac{S^{-1}G\big(e_j+O(\sqrt\epsilon)\big)}{|x|}=\frac{(\partial_jS^{-1})G\dot\gamma_\epsilon(1)}{|x|}+\frac{S^{-1}\nabla G\cdot e_j\dot\gamma_\epsilon(1)}{|x|}.\tag{5.20}$$

The ith component of the right hand side of (5.20) is of the form

$$-\frac{\hat x_i\hat x_j}{|x|^2}+|x|^{-2}\big((P_\perp-P_\perp GP_\perp)e_j\big)_i+\frac{O(\sqrt\epsilon)}{|x|^2}.$$

Obviously

$$\frac{\big(S^{-1}Ge_j\big)_i}{|x|}=\frac{\big(Ge_j\big)_i}{|x|^2}+\frac{O(\sqrt\epsilon)}{|x|^2}.$$

By adding these expressions we obtain that

$$\frac{\partial_{ij}S}{|x|}=-\frac{\hat x_i\hat x_j-\delta_{ij}+O(\sqrt\epsilon)}{|x|^2}.\tag{5.21}$$

We combine (5.19) and (5.21) and conclude (1.15c). $\square$

Remark 5.1. An analogue of (5.5) was proved by a different technique in [Ba] (see [Ba, Lemma 3]) which does not work in the present context. Note that in [Ba] also an equation like (5.9) is used, and in fact it is used to prove the analogue of (5.5). However in our case there is no smallness of the factor $G^{-1}P_\perp\nabla G\cdot\dot\gamma_\epsilon P_\perp$ and consequently the technique in [Ba] is not applicable. Note also that [Ba, Lemma 4] is an analogue of (5.7).

6. Proof of Theorem 1.6

In this section we need the Sobolev spaces $\mathcal{H}^p := W_0^{1,p}(0,1)^d$, $1<p<\infty$, consisting of absolutely continuous functions $h:[0,1]\to\mathbb{R}^d$ vanishing at the endpoints and having $\dot h\in L^p(0,1)^d = L^p(]0,1[,\mathbb{R}^d)$ (throughout this section we use the notation L^p for this vector-valued L^p space). The space $\mathcal{H}^p$ is equipped with the norm

$$\|h\|_{\mathcal{H}^p} = \|\dot h\|_p = \Big(\int_0^1 |\dot h(s)|^p ds\Big)^{1/p}. \tag{6.1}$$

The first goal is to find a substitute for the positivity bound (4.23) used in the proof of Theorem 1.4. We shall use the observation, cf. the proof of Lemma 4.1, that

$$\langle\partial_\kappa^2 E(x,0)h_1,h_2\rangle = \int_0^1 2s^2\big\{\big(s^{-1}h_1(s)\big)^{\dot{}}\,G(sx)\big(s^{-1}h_2(s)\big)^{\dot{}}\big\}ds. \tag{6.2}$$

Motivated by (6.2) we develop some functional analysis which then will be applied to an extension/modification for perturbed geodesics.

6.1. **Hardy type bounds and duality theory.**

Lemma 6.1. *For all $p\in]1,\infty[$ there exists $A_p>0$ such that*

$$\|h(\cdot)/s\|_p = \Big(\int_0^1 |h(s)/s|^p ds\Big)^{1/p} \le A_p\|h\|_{\mathcal{H}^p} \text{ for all } h\in\mathcal{H}^p. \tag{6.3}$$

Proof. We refer to [HLP, Theorem 327]; the bound is valid for $A_p = p/(p-1)$. □

Lemma 6.2. *For all $p\in]1,\infty[$ there exists $B_p>0$ such that*

$$\|h\|_{\mathcal{H}^p} \le B_p\|\dot h(\cdot) - h(\cdot)/s\|_p \text{ for all } h\in\mathcal{H}^p. \tag{6.4}$$

Proof. Consider the linear maps

$$L^p(0,1)^d \ni f\to Sf;\ (Sf)(t) = t^{-1}\int_0^t f(s)ds. \tag{6.5a}$$

$$L^p(0,1)^d \ni f\to Tf;\ (Tf)(t) = -t\int_t^1 s^{-1}f(s)ds. \tag{6.5b}$$

Note that S is bounded on L^p with $\|S\|_{\mathcal{B}(L^p)}\le A_p$, cf. Lemma 6.1 and its proof. Next note

$$\tfrac{d}{dt}(Tf)(t) = t^{-1}(Tf)(t) + f(t), \tag{6.6}$$

and recall the standard fact

$$L^p(I,\mathcal{G}) = (L^q(I,\mathcal{G})^* \text{ if } q^{-1} = 1-p^{-1}; \tag{6.7}$$

in our case $I=]0,1[$ and $\mathcal{G}$ is the Hilbert space $\mathbb{R}^d$. In terms of (6.7) we can rewrite (6.6) as

$$\dot T := \tfrac{d}{dt}T = -S^* + I,$$

and since $S \in \mathcal{B}(L^q)$ we conclude that $\dot T \in \mathcal{B}(L^p)$.

Finally noting that for any $h \in \mathcal{H}^p$ we can write $h = Tf$ where $f(s) = \dot h(s) - h(s)/s$ we obtain (6.4) with $B_p = A_q + 1$. $\square$

Remark 6.3. By the estimate (due to the Hölder inequality)

$$\Big| \int_t^1 s^{-1} f(s) ds \Big| \le t^{-1/p} (p-1)^{1-1/p} \|f\|_p,$$

it follows from the proof of Lemma 6.2 that the operator T of (6.5b) is in $\mathcal{B}(L^p, \mathcal{H}^p)$ with norm $\|\dot T\|_{\mathcal{B}(L^p)} \le B_p$. In fact (using here also Lemma 6.1) $T : L^p \to \mathcal{H}^p$ is a linear homeomorphism.

Lemma 6.4. *Suppose G is a given continuous function, $[0,1] \to G(s) \in \mathcal{S}_d(\mathbb{R})$, using here notation of Subsection 1.1. Suppose there are constants $a, b > 0$ such that*

$$a|y|^2 \le yG(s)y \le b|y|^2, \qquad y \in \mathbb{R}^d \text{ and } s \in [0,1]. \tag{6.8}$$

Introduce for $h \in \mathcal{H}^p$ and $g \in \mathcal{H}^q$, $q^{-1} + p^{-1} = 1$, the pairing

$$[h, g] = \int_0^1 2s^2 \big\{ \big(s^{-1} h(s)\big)^{\cdot} G(s) \big(s^{-1} g(s)\big)^{\cdot} \big\} ds, \tag{6.9a}$$

and the associated quantity

$$\|h\|_{\mathcal{H}^p, G} = \sup_{\|g\|_{\mathcal{H}^q} \le 1} |[h, g]|. \tag{6.9b}$$

This quantity $\|\cdot\|_{\mathcal{H}^p, G}$ is a norm on $\mathcal{H}^p$ and

$$\begin{gathered} C_1^{-1} \|h\|_{\mathcal{H}^p, G} \le \|h\|_{\mathcal{H}^p} \le C_2 \|h\|_{\mathcal{H}^p, G}; \\ C_1 = 2b(1 + A_q)(1 + A_p),\ C_2 = (2a)^{-1} B_q B_p. \end{gathered} \tag{6.10}$$

Here C_1 and C_2 are given in terms of the constants of Lemmas 6.1-6.2 and (6.8).

Finally $\big(\mathcal{H}^p\big)^* = \mathcal{H}^q$ in the sense given by the pairing (6.9a). This means that for all $l \in \big(\mathcal{H}^p\big)^*$ there exists a unique $g \in \mathcal{H}^q$ such that

$$l(h) = [h, g] \text{ for all } h \in \mathcal{H}^p, \tag{6.11}$$

and vica versa any $g \in \mathcal{H}^q$ defines an element $l \in \big(\mathcal{H}^p\big)^*$ by (6.11). Moreover the identification is a linear homeomorphism.

Proof. Notice that indeed the pairing (6.9a) is well-defined due to the Hölder and Minkowski inequalities and Lemma 6.1. Using these bounds we also get the first estimate of (6.10).

To prove the second estimate of (6.10) we first use Lemma 6.2 and (6.7) to estimate

$$\begin{aligned} \|h\|_{\mathcal{H}^p} &\le B_p \sup_{\|2Gf\|_{L^q} \le 1} \Big| \int_0^1 s \big(s^{-1} h(s)\big)^{\cdot} 2G(s) f(s) ds \Big| \\ &\le B_p \sup_{\|f\|_{L^q} \le (2a)^{-1}} \Big| \int_0^1 s \big(s^{-1} h(s)\big)^{\cdot} 2G(s) f(s) ds \Big|. \end{aligned}$$

Next we introduce for any $f \in L^q$ the function $g = Tf$ which according to Remark 6.3 is an element of $\mathcal{H}^q$. We obtain

$$\begin{aligned}\|h\|_{\mathcal{H}^p} &\le B_p \sup_{\|f\|_{L^q}\le(2a)^{-1},\, g=Tf} \Big|\int_0^1 s\big(s^{-1}h(s)\big)^{\cdot} 2G(s)s\big(s^{-1}g(s)\big)^{\cdot} ds\Big| \\ &\le B_p \sup_{\|g\|_{\mathcal{H}^q}\le(2a)^{-1}B_q} \Big|\int_0^1 s\big(s^{-1}h(s)\big)^{\cdot} 2G(s)s\big(s^{-1}g(s)\big)^{\cdot} ds\Big| \\ &= B_p(2a)^{-1}B_q\,\|h\|_{\mathcal{H}^p,G}.\end{aligned}$$

We have proved (6.10).

The identification asserted next follows similarly from (6.7) and Remark 6.3. The bi-continuity is a consequence of (6.10). ☐

Corollary 6.5. *Let G be given as in Lemma 6.4, and define $P \in \mathcal{B}(\mathcal{H})$, $\mathcal{H} = \mathcal{H}^2$, by*

$$\langle Ph_1, h_2\rangle = [h_1, h_2];\ h_1, h_2 \in \mathcal{H}.$$

Let C_1 and C_2 be the constants in (6.10).

For all $p \in [2,\infty[$ the operator $P \in \mathcal{B}\big(\mathcal{H}^p\big)$ and it obeys

$$\|P\|_{\mathcal{B}(\mathcal{H}^p)} \le 2C_1. \tag{6.12a}$$

In particular (with $q \in]1,2]$ being the conjugate exponent)

$$|\langle Ph, g\rangle| \le 2C_1\|h\|_{\mathcal{H}^p}\,\|g\|_{\mathcal{H}^q} \text{ for all } h \in \mathcal{H}^p \text{ and } g \in \mathcal{H}^2. \tag{6.12b}$$

Conversely, if

$$h \in \mathcal{H}^p \text{ and } |\langle Ph, g\rangle| \le c\|g\|_{\mathcal{H}^q} \text{ for all } g \in \mathcal{H}^2, \tag{6.13a}$$

then

$$\|h\|_{\mathcal{H}^p} \le cC_2. \tag{6.13b}$$

Proof. We show that $Ph \in \mathcal{H}^p$ for all $h \in \mathcal{H}^p$ if $p \in [2,\infty[$: Fix any such p and such h, and let $q \in]1,2]$ be the conjugate exponent. Then $Ph \in \mathcal{H}^p$ if the expression

$$\|Ph\|_{\mathcal{H}^p} = \sup_{\|f\|_{L^q}\le 1, f\in L^2} \Big|\int_0^1 (Ph)^{\cdot}(s)f(s)ds\Big| < \infty.$$

We introduce the map

$$L^q \ni f \to g = Rf;\ (Rf)(t) = \int_0^t \Big(f(s) - \int_0^1 f(l)dl\Big)ds. \tag{6.14}$$

Clearly R maps into $\mathcal{H}^q$ and in fact (due to the Hölder inequality)

$$\|R\|_{\mathcal{B}(L^q,\mathcal{H}^q)} \le 2.$$

Whence

$$\begin{aligned}\|Ph\|_{\mathcal{H}^p} &= \sup_{\|f\|_{L^q}\le 1, f\in L^2} \Big|\int_0^1 (Ph)^{\cdot}(s)(Rf)^{\cdot}(s)ds\Big| \\ &\le \sup_{\|g\|_{\mathcal{H}^q}\le 2, g\in\mathcal{H}^2} \Big|\int_0^1 (Ph)^{\cdot}(s)\dot g(s)ds\Big| \\ &\le \sup_{\|g\|_{\mathcal{H}^q}\le 2, g\in\mathcal{H}^2} |[h,g]| \\ &\le 2C_1\|h\|_{\mathcal{H}^p}.\end{aligned}$$

So indeed $Ph \in \mathcal{H}^p$ and the estimates show (6.12a).

Clearly (6.12b) follows from the Hölder inequality and (6.12a).

As for (6.13b) we note that by a density argument and (6.13a)

$$\|h\|_{\mathcal{H}^p,G} = \sup_{\|g\|_{\mathcal{H}^q}\leq 1, g\in\mathcal{H}^2} |[h,g]| = \sup_{\|g\|_{\mathcal{H}^q}\leq 1, g\in\mathcal{H}^2} |\langle Ph, g\rangle| \leq c.$$

Whence we can conclude by (6.10). □

Remark. In our application, see Subsection 6.2, $G(s)$ will be a composition of a metric in $\mathcal{O}$ and a perturbed geodesic (as in (6.2) for the unperturbed case). We shall conclude (6.13b) for functions in question by verifying (6.13a).

For completeness of presentation let us note the following additional property of the operator P (shown to be in $\mathcal{B}(\mathcal{H}^p)$): For all $p \in [2,\infty[$ in fact P is a linear homeomorphism on $\mathcal{H}^p$. Outline of a proof: We verify the condition (6.13a) using the Hölder inequality and deduce from the conclusion (6.13b) that

$$\|h\|_{\mathcal{H}^p} \leq C_2\|Ph\|_{\mathcal{H}^p} \text{ for all } h \in \mathcal{H}^p.$$

Whence $P : \mathcal{H}^p \to \mathcal{H}^p$ is injective and has dense range. It remains to show that its range is dense. For that we note that

$$\langle h, g\rangle = \int_0^1 \dot{h}\dot{g}\,ds,\ h \in \mathcal{H}^p \text{ and } g \in \mathcal{H}^q,$$

defines a pairing giving another identification $(\mathcal{H}^p)^* = \mathcal{H}^q$ (use the first step of the proof of Corollary 6.5 and Hahn-Banach theory). Whence if the range is not dense we can pick $0 \neq g \in \mathcal{H}^q$ such that $\langle Ph, g\rangle = 0$ for all $h \in \mathcal{H}^p$, violating that $\langle Ph, g\rangle = [h, g]$ and the last part of Lemma 6.4.

6.2. **Bounds of $\partial^\alpha\kappa$ in $\mathcal{H}^p$, $p \in [2,\infty[$.** We shall improve on Proposition 3.1 iii) in the case of a metric $\tilde{G} \in \mathcal{M}$ close to a given $G \in \mathcal{O}$ (as in Theorem 1.4 ii)). We shall use the convention used in Subsection 4.2 and write in terms of an "order parameter" $\tilde{G} = G_\epsilon$ if $\|G_\epsilon - G\|_l \leq \epsilon$. Following the proof of Proposition 3.1 iii) we can for $1 \leq |\alpha| \leq l-1$ represent the quantity

$$-\langle\partial^2_\kappa E_\epsilon(x,\kappa)\partial^\alpha\kappa, h\rangle$$

as a sum of terms involving derivatives $\partial^\beta\kappa$ where $|\beta| \leq |\alpha| - 1$. Using a new induction hypothesis we shall estimate these terms individually improving the corresponding bounds of the proof of Proposition 3.1 iii). On the other hand we can write

$$\partial^2_\kappa E_\epsilon(x,\kappa) = P + R; \tag{6.15a}$$

$$\langle Ph_1, h_2\rangle = [h_1, h_2] := \int_0^1 2s^2\big\{\big(s^{-1}h_1(s)\big)^{\cdot} G(\gamma_\epsilon(s))\big(s^{-1}h_2(s)\big)^{\cdot}\big\}ds. \tag{6.15b}$$

The second term in (6.15a) can be estimated by the following

Lemma 6.6. *Let $p \in [2,\infty[$ and q be the conjugate exponent. There exists $\epsilon_0 > 0$ and $C > 0$ such that for all $x \in \mathbb{R}^d$ and for $\|G_\epsilon - G\|_l \leq \epsilon \leq \epsilon_0$*

$$|\langle Rh_1, h_2\rangle| \leq \sqrt{\epsilon}C\|h_1\|_{\mathcal{H}^p}\|h_2\|_{\mathcal{H}^q} \text{ for all } h_1 \in \mathcal{H}_p \text{ and } h_2 \in \mathcal{H}_2. \tag{6.16}$$

Proof. The result follows from (4.31) using (4.25), (4.27), the Hölder inequality and Lemma 6.1. □

Lemma 6.7. *Let p, q, $\epsilon_0 > 0$ and $C > 0$ be given as in lemma 6.6. Suppose $1 \le |\alpha| \le l-1$ and that for some constant $C_\alpha > 0$ independent of $x \in \mathbb{R}^d$ and $\epsilon \in [0, \epsilon_0]$*

$$|\langle \partial^2_\kappa E_\epsilon(x,\kappa)\partial^\alpha \kappa, h\rangle| \le C_\alpha \langle x\rangle^{1-|\alpha|}\|h\|_{\mathcal{H}^q} \text{ for all } h \in \mathcal{H}^2. \tag{6.17}$$

Let $a, b > 0$ be the constants from (1.3) *determined by the metric G and let C_2 be the constant from* (6.10) *(fixed in terms of a and p). Then*

$$\left(1 - \sqrt{\epsilon}CC_2\right)\|\partial^\alpha \kappa\|_{\mathcal{H}^p} \le C_\alpha C_2 \langle x\rangle^{1-|\alpha|}. \tag{6.18}$$

Proof. Clearly (6.8) holds for the example $G(\gamma_\epsilon(\cdot))$ used in (6.15b). From (6.15a), (6.15b) and Lemma 6.6 we deduce (6.13a) with

$$c = C_\alpha\langle x\rangle^{1-|\alpha|} + \sqrt{\epsilon}C\|\partial^\alpha\kappa\|_{\mathcal{H}^p},$$

and we obtain from (6.13b) that

$$\|\partial^\alpha\kappa\|_{\mathcal{H}^p} \le \left(C_\alpha\langle x\rangle^{1-|\alpha|} + \sqrt{\epsilon}C\|\partial^\alpha\kappa\|_{\mathcal{H}^p}\right)C_2$$

yielding (6.18) by subtraction. ☐

Proposition 6.8. *Let $p \in [2,\infty[$. There exist $\epsilon_0 > 0$ and $C_p > 0$ such that for $x \in \mathbb{R}^d$, $\|G_\epsilon - G\|_l \le \epsilon \le \epsilon_0$ and $1 \le |\alpha| \le l-1$*

$$\|\partial^\alpha\kappa\|_{\mathcal{H}^p} \le C_p\langle x\rangle^{1-|\alpha|}. \tag{6.19}$$

Proof. Using (6.18) for all sufficiently small ϵ we only need to demonstrate (6.17). So let us verify (6.17) for the case $|\alpha| = 1$: We mimic Step I in the proof of Theorem 1.4 ii) using the Hölder inequality and Lemma 6.1 (to replace the Cauchy Schwarz and Hardy inequalities, respectively). This yields (6.17) for $|\alpha| = 1$, in fact with any extra factor $\sqrt{\epsilon}$. We have shown the proposition for the case $l = 2$.

Next let us suppose $l \ge 3$ and that we know the bounds (6.19) for all $p \in [2,\infty[$ and for $|\alpha| \in [1, n-1]$ where $l - 1 \ge n \ge 2$ (notice that $\epsilon_0 > 0$ may depend on p). Let $p \in [2,\infty[$ and α with $|\alpha| = n$ be given. The proof is complete if we can show the existence of $\epsilon_0 > 0$ such that (6.17) holds for the given p and α provided $\|G_\epsilon - G\|_l \le \epsilon \le \epsilon_0$. For that we need to modify the proof of Proposition 3.1 iii). Again we can assume that $|x| \ge 1$. We shall use the induction hypothesis for

$$p \to \tilde{p} := 4np. \tag{6.20}$$

This particular choice fixes some $\epsilon_0 > 0$ that we claim indeed works for (6.17) (it is not claimed to be an "optimal" choice of $\tilde{p}$ although growth in n is indispensable). The Hardy inequality (3.11a) is replaced by Lemma 6.1 and (3.11b) by

$$|\tilde{h}(s)| \le s^{1-1/\tilde{p}}\|\tilde{h}\|_{\mathcal{H}^{\tilde{p}}}; \tag{6.21}$$

here we shall use the $\tilde{p}$ of (6.20). We shall also need the generalized Hölder inequality

$$\int_0^1 |f_1(s)|\cdots|f_m(s)|\,ds \le \|f_1\|_{p_1}\cdots\|f_m\|_{p_m} \text{ for } 1/p_1 + 1/p_m \le 1. \tag{6.22}$$

Finally we shall use the following special case of (3.10)

$$|\partial^\eta g_{ij}| \le C|sx|^{-|\eta|}; \tag{6.23}$$

here and henceforth we omit the subscript ϵ (obviously C is independent of ϵ). Note that (6.23) is the best possible bound at infinity. The main issue compared to the proof of Proposition 3.1 iii) is that the improved pointwise bound (6.21), used to $\tilde{h} = \partial^{\beta_j}\kappa$, compensates for the worse singularity at $s = 0$ when using (6.23).

Now let us look at some details: There are cases A), B) and C) defined as in the proof of Proposition 3.1 iii) (and we treat only $k \geq 1$).

For the case A) we need (6.23) with $|\eta| = |\zeta| + k - 1$, and we consider subcases Ai)–Aiii) defined as before.

Case Ai): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^\zeta \partial_{\kappa(s)}^\omega G(sx + \kappa(s))$; $|\omega| = k - 1$. Suppose first that $k \geq 2$ and $i, j \leq k$. Then we shall use (6.22) with $m = 4$, $p_1 < \tilde{p}/n$, $p_2 = p_3 = \tilde{p}$ and $p_4 = q$, $f_1(s) = s^{-n/\tilde{p}}$, $f_2 = \dot{h}_i$, $f_3 = \dot{h}_j$ and $f_4(s) = h(s)/s$. Note that this is legitimate if $\tilde{p}/n - p_1 > 0$ is sufficiently small; it yields the extra factor $s^{1+n/\tilde{p}}$ appearing below. Upon using the pointwise bound (6.21) for the remaining $k - 2$ factors of components of $h_\bullet$'s, (6.23) and the fact (since $k \leq n$) that

$$s^{-|\eta|} s^{|\zeta|} s^{(k-2)\big(1-1/\tilde{p}\big)} s^{1+n/\tilde{p}} \leq 1,$$

we obtain the bound

$$|(F_{\zeta,k}; h_1, \dots, h_{k+1})| \leq C|x|^{-|\eta|} \prod_{m=1}^{k} \|h_m\|_{\mathcal{H}^{\tilde{p}}} \|h\|_{\mathcal{H}^q}. \tag{6.24}$$

By the induction hypothesis

$$\prod_{m=1}^{k} \|h_m\|_{\mathcal{H}^{\tilde{p}}} \leq C\langle x\rangle^{k-\sum|\beta_m|} = C\langle x\rangle^{k-(n-|\zeta|)},$$

which together with (6.24) yields

$$|(F_{\zeta,k}; h_1, \dots, h_{k+1})| \leq C\langle x\rangle^{-|\eta|}\langle x\rangle^{k-(n-|\zeta|)}\|h\|_{\mathcal{H}^q} = C\langle x\rangle^{1-n}\|h\|_{\mathcal{H}^q}. \tag{6.25}$$

Suppose next that $j = k + 1$. Then we apply (6.22) with $m = 3$, $p_1 < \tilde{p}/n$, $p_2 = \tilde{p}$ and $p_3 = q$, $f_1(s) = s^{-n/\tilde{p}}$, $f_2 = \dot{h}_i$ and $f_3 = \dot{h}$ yielding an extra factor $s^{n/\tilde{p}}$. Upon using the pointwise bound (6.21) for the remaining $k - 1$ factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|} s^{|\zeta|} s^{(k-1)(1-1/\tilde{p})} s^{n/\tilde{p}} \leq 1,$$

we obtain again (6.24) and whence (6.25).

Case Aii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_1|}\partial_x^{\zeta_1}\partial_{\kappa(s)}^\omega G(sx+\kappa(s))$; $|\zeta_1| = |\zeta| - 1$, $|\omega| = k$. Suppose first that $j \leq k$. Then we apply (6.22) with $m = 3$, $p_1 < \tilde{p}/n$, $p_2 = \tilde{p}$ and $p_3 = q$, $f_1(s) = s^{-n/\tilde{p}}$, $f_2 = \dot{h}_j$ and $f_3(s) = h(s)/s$ yielding an extra factor $s^{1+n/\tilde{p}}$. Upon using (6.21) for the remaining $k - 1$ factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|} s^{|\zeta_1|} s^{(k-1)(1-1/\tilde{p})} s^{1+n/\tilde{p}} \leq 1,$$

we obtain again (6.24) and whence (6.25).

Suppose next that $j = k + 1$. Then we apply (6.22) with $m = 2$, $p_1 < \tilde{p}/n$, $p_2 = q$, $f_1(s) = s^{-n/\tilde{p}}$, $f_2 = \dot{h}$ yielding an extra factor $s^{n/\tilde{p}}$. Upon using (6.21) for the remaining k factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|} s^{|\zeta_1|} s^{k(1-1/\tilde{p})} s^{n/\tilde{p}} \leq 1,$$

we conclude as before.

Case Aiii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_2|}\partial_x^{\zeta_2}\partial_{\kappa(s)}^\omega G(sx + \kappa(s))$; $|\zeta_2| = |\zeta| - 2$, $|\omega| = k + 1$. We apply (6.22) with $m = 2$, $p_1 < \tilde{p}/n$, $p_2 = q$, $f_1(s) = s^{-n/\tilde{p}}$ and $f_2(s) = h(s)/s$

yielding an extra factor $s^{1+n/\tilde p}$. Upon using (6.21) for the remaining k factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|}s^{|\zeta_2|}s^{k(1-1/\tilde p)}s^{1+n/\tilde p}\le 1,$$

we conclude as before.

For the case **B)** we need (6.23) with $|\eta| = |\zeta| + k$, and we consider subcases **Bi)**–**Bii)** defined as before.

Case Bi): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^\zeta\partial^\omega_{\kappa(s)}G(sx+\kappa(s))$; $|\omega| = k$. Suppose first that $j\le k$. Then we apply (6.22) with $m = 3$, $p_1 < \tilde p/n$, $p_2 = \tilde p$ and $p_3 = q$, $f_1(s) = s^{-n/\tilde p}$, $f_2 = \dot h_j$ and $f_3(s) = h(s)/s$ yielding an extra factor $s^{1+n/\tilde p}$. Upon using (6.21) for the remaining $k-1$ factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|}s^{|\zeta|}s^{(k-1)(1-1/\tilde p)}s^{1+n/\tilde p}\le 1,$$

we obtain the bound

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})|\le C|x|^1|x|^{-|\eta|}\prod_{m=1}^k\|h_m\|_{\mathcal H^{\tilde p}}\|h\|_{\mathcal H^q}. \tag{6.26}$$

By the induction hypothesis

$$\prod_{m=1}^k\|h_m\|_{\mathcal H^{\tilde p}}\le C\langle x\rangle^{k-\sum|\beta_m|} = C\langle x\rangle^{k-(n-|\zeta|)},$$

which together with (6.26) yields

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})|\le C\langle x\rangle^{1-|\eta|}\langle x\rangle^{k-(n-|\zeta|)}\|h\|_{\mathcal H^q} = C\langle x\rangle^{1-n}\|h\|_{\mathcal H^q}. \tag{6.27}$$

Suppose next that $j = k+1$. Then we apply (6.22) with $m = 2$, $p_1 < \tilde p/n$, $p_2 = q$, $f_1(s) = s^{-n/\tilde p}$, $f_2 = \dot h$ yielding an extra factor $s^{n/\tilde p}$. Upon using (6.21) for the remaining k factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|}s^{|\zeta|}s^{k(1-1/\tilde p)}s^{n/\tilde p}\le 1,$$

we conclude as before.

Case Bii): $\partial^\eta G(\gamma(s)) = s^{-|\zeta_1|}\partial_x^{\zeta_1}\partial^\omega_{\kappa(s)}G(sx+\kappa(s))$; $|\zeta_1| = |\zeta|-1$, $|\omega| = k+1$. We apply (6.22) with $m = 2$, $p_1 < \tilde p/n$, $p_2 = q$, $f_1(s) = s^{-n/\tilde p}$ and $f_2(s) = h(s)/s$ yielding an extra factor $s^{1+n/\tilde p}$. Upon using (6.21) for the remaining k factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|}s^{|\zeta_1|}s^{k(1-1/\tilde p)}s^{1+n/\tilde p}\le 1,$$

we conclude as before.

For the case C) we need (6.23) with $|\eta| = |\zeta| + k + 1$.

Case C): $\partial^\eta G(\gamma(s)) = s^{-|\zeta|}\partial_x^\zeta\partial^\omega_{\kappa(s)}G(sx+\kappa(s))$; $|\omega| = k+1$.

We apply (6.22) with $m = 2$, $p_1 < \tilde p/n$, $p_2 = q$, $f_1(s) = s^{-n/\tilde p}$ and $f_2(s) = h(s)/s$ yielding an extra factor $s^{1+n/\tilde p}$. Upon using (6.21) for the remaining k factors of components of $h_\bullet$'s, (6.23) and the fact that

$$s^{-|\eta|}s^{|\zeta|}s^{k(1-1/\tilde p)}s^{1+n/\tilde p}\le 1,$$

we obtain the bound

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})| \le C|x|^2|x|^{-|\eta|}\prod_{m=1}^{k}\|h_m\|\|_{\mathcal{H}^{\tilde p}}\|h\|_{\mathcal{H}^q}. \tag{6.28}$$

By the induction hypothesis

$$\prod_{m=1}^{k}\|h_m\|_{\mathcal{H}^{\tilde p}} \le C\langle x\rangle^{k-\sum|\beta_m|} = C\langle x\rangle^{k-(n-|\zeta|)},$$

which together with (6.28) yields

$$|(F_{\zeta,k};h_1,\dots,h_{k+1})| \le C\langle x\rangle^{2-|\eta|}\langle x\rangle^{k-(n-|\zeta|)}\|h\|_{\mathcal{H}^q} = C\langle x\rangle^{1-n}\|h\|_{\mathcal{H}^q}. \tag{6.29}$$

□

Remarks. 1) As noticed in the beginning of the above proof, using the proof of Theorem 1.4 ii) we can improve (6.19) for $|\alpha| = 1$ as

$$\|\partial_x^\alpha\kappa\|_{\mathcal{H}^p} \le \sqrt{\epsilon}C_p \text{ for } |\alpha| = 1. \tag{6.30}$$

2) By integrating (6.30) we obtain the bound

$$\|\kappa\|_{\mathcal{H}^p} \le \sqrt{\epsilon}C_p\langle x\rangle. \tag{6.31}$$

From the method of proof we have $C_p \to \infty$ as $p \to \infty$. We remark that this feature is not an artifact of the proof. In fact for the example in Subsection 7.3 the geodesics emanating from 0 are rotating like logarithmic spirals showing that (6.31) is false for $p = \infty$ (here by definition $\mathcal{H}^\infty = W_0^{1,\infty}(0,1)^d$). This partly explains why we worked with (6.15b) rather than the ϵ-independent pairing (6.2).

Proof of Theorem 1.6. We mimic the proof of Proposition 3.1 iv). Indeed combining (3.7) and the bounds (6.19) and (6.31) (with $p = 2$) we show the following improvement of (3.21)

$$|\partial^\alpha\dot\kappa(1)| \le C\langle x\rangle^{1-|\alpha|} \text{ for all } |\alpha| \le l-1.$$

Next we use again (3.5) and obtain (1.17a). Obviously we can similarly use (3.5) to prove (1.17b). □

7. Examples

In this section we present examples of metrics in the class $\mathcal{O}$. Two of the examples are parameter-depending and are constructed by the exponential mapping from metrics that are not of order zero.

7.1. Decaying potentials.

Let V be a radial function of class C^l on $\mathbb{R}^d$, $l,d \ge 2$, for which there are constants $a > 0$, $A > 0$, $\mu \in]0,2[$ and $\sigma \in]0,2]$ such that

$$-A\langle x\rangle^{-\mu} \le V(x) \le -a\langle x\rangle^{-\mu}, \tag{7.1a}$$

$$x\cdot\nabla V(x) + 2V(x) \le \sigma V(x), \tag{7.1b}$$

$$\partial^\alpha V(x) = O\big(\langle x\rangle^{-(\mu+|\alpha|)}\big) \text{ for } |\alpha| \le l. \tag{7.1c}$$

Consider the functional $J : \mathbb{R}^d \times \mathcal{H} \to \mathbb{R}$ given by

$$J(x,\kappa) = \int_0^1 K(y(s))|\dot y(s)|^2 ds, \tag{7.2}$$

where $y(s) = sx + \kappa(s)$ and $K(x) = 2(\lambda - V(x))$ for $\lambda \geq 0$, and consider the positive solution $S(x)$ to the eikonal equation

$$|\nabla S(x)|^2 = K(x) \text{ for } x \in \mathbb{R}^d \setminus \{0\}, \tag{7.3}$$

defined by

$$S(r) = \int_0^r \sqrt{K(\tau)}d\tau,$$

or alternatively by

$$S^2(x) = \inf\{J(x,\kappa) : \kappa \in \mathcal{H}\}.$$

We introduce the diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\Phi(x) = r(|x|)\hat{x} = r(|x|)\frac{x}{|x|}, \tag{7.4}$$

where r is the inverse of the function

$$\rho(r) = S(r) = \int_0^r \sqrt{K(\tau)}d\tau.$$

Note that $\Phi(t\sqrt{K(0)}x) = \exp_0(tx)$, i.e. the exponential mapping at zero for the metric $g_{ij} = K\delta_{ij}$ of (7.2). A short calculation shows that if $z(s) = \Phi(y(s))$ and $\omega(s) = y(s)/|y(s)|$ then

$$\begin{aligned}\dot{z} &= \frac{1}{\sqrt{K(r(|y|))}}\dot{y}\cdot\omega\omega + \frac{r(|y|)}{|y|}(\dot{y} - \dot{y}\cdot\omega\omega) \\ &= \frac{1}{\sqrt{K(r(|y|))}}P\dot{y} + \frac{r(|y|)}{|y|}P_\perp\dot{y}\end{aligned}$$

where P is the projection parallel to ω and $P_\perp = I - P$ the projection onto $\{\omega\}^\perp$. Therefore

$$\begin{aligned}K(z)|\dot{z}|^2 &= |P\dot{y}|^2 + f^2(|y|)|P_\perp\dot{y}|^2 \\ &= \dot{y}G(y)\dot{y}\end{aligned}$$

where

$$G(y) = P + f^2(|y|)P_\perp \tag{7.5a}$$

and

$$f(\rho) = \frac{\sqrt{K(r(\rho))}r(\rho)}{\rho} = \left(\int_0^1 \sqrt{\frac{\lambda - V(sr(\rho))}{\lambda - V(r(\rho))}}ds\right)^{-1}. \tag{7.5b}$$

Thus, if $y(s) = sx + \kappa(s)$ and $z(s) = \Phi(y(s)) = s\Phi(x) + h(s)$, with $\kappa, h \in \mathcal{H}$, then

$$J(\Phi(x), h) = \int_0^1 \dot{y}(s)G(y(s))\dot{y}(s)ds =: E(x,\kappa)$$

and therefore

$$\inf\{E(x,\kappa) : \kappa \in \mathcal{H}\} = \inf\{J(\Phi(x), h) : h \in \mathcal{H}\} = S^2(\Phi(x)).$$

Now we show that G satisfies (1.3) with constants a and b independent of $\lambda \geq 0$. It suffices to prove there exist $c > 0$ and $C > 0$ such that

$$c \leq \int_0^1 \sqrt{\frac{\lambda - V(sr(\rho))}{\lambda - V(r(\rho))}}ds \leq C,$$

for all $\lambda \geq 0$ and $r \geq 0$. Note first that from (7.1a) we have

$$\frac{\lambda + a\langle sr\rangle^{-\mu}}{\lambda + A\langle r\rangle^{-\mu}} \leq \frac{\lambda - V(sr)}{\lambda - V(r)} \leq \frac{\lambda + A\langle sr\rangle^{-\mu}}{\lambda + a\langle r\rangle^{-\mu}}$$

The lower bound follow from the fact that

$$\frac{\lambda + a\langle sr\rangle^{-\mu}}{\lambda + A\langle r\rangle^{-\mu}} \geq \frac{\lambda + a\langle r\rangle^{-\mu}}{\lambda + A\langle r\rangle^{-\mu}} \geq \frac{a}{A}, \tag{7.6a}$$

for $\lambda \geq 0$, $r \geq 0$ and $s \in [0, 1]$. Next we note that

$$\frac{\lambda + A\langle sr\rangle^{-\mu}}{\lambda + a\langle r\rangle^{-\mu}} \leq \frac{A}{a} \frac{\langle r\rangle^{\mu}}{\langle sr\rangle^{\mu}}, \tag{7.6b}$$

and therefore

$$\begin{aligned}\int_0^1 \sqrt{\frac{\lambda + A\langle sr\rangle^{-\mu}}{\lambda + a\langle r\rangle^{-\mu}}} ds &\leq \sqrt{\frac{A}{a}} \langle r\rangle^{\mu/2} \int_0^1 \frac{ds}{(1 + s^2 r^2)^{\mu/4}} \\ &= \sqrt{\frac{A}{a}} \frac{\langle r\rangle^{\mu/2}}{r} \int_0^r \frac{du}{(1 + u^2)^{\mu/4}}.\end{aligned}$$

The upper bound is obtained from the fact that the function

$$\phi(r) = \frac{\langle r\rangle^{\mu/2}}{r} \int_0^r \frac{du}{(1 + u^2)^{\mu/4}}$$

is continuous for $r > 0$ and satisfies

$$\lim_{r\to 0} \phi(r) = 1, \qquad \text{and} \qquad \lim_{r\to\infty} \phi(r) = 2/(2 - \mu).$$

Thus, since it can easily be verified that G satisfies (1.2), it follows that indeed G satisfies (1.3). From from (7.5a) it follows that G satisfies (1.12). A computation using (7.1b) shows the bound (1.13) with any $\bar{c} \leq \frac{\sigma}{2}/\sup f$, whence $G \in \mathcal{O}$.

Consequently, due to Theorem 1.4, if a real C^l-potential on $\mathbb{R}^d$, say V_ϵ, is sufficiently close to the radial potential V discussed above in the sense that for a sufficiently small $\epsilon > 0$

$$\|V_\epsilon - V\|_l := \sup_{|\alpha|\leq l} \sup_x \langle x\rangle^{|\alpha|+\mu} |\partial^\alpha\big(V_\epsilon(x) - V(|x|)\big)| \leq \epsilon, \tag{7.7}$$

then there exists a C^l-solution S_ϵ to the eikonal equation (7.3) with $V \to V_\epsilon$. Indeed writing the solution from Theorem 1.4 defined in terms of the perturbed metric, say G_ϵ, as $\tilde{S}_\epsilon$ (in the coordinates $y = \Phi^{-1}(z)$) we have $S_\epsilon(x) = \tilde{S}_\epsilon(\Phi^{-1}(x))$ and we can use the estimates of Theorem 1.4 to compare the derivatives $\partial^\alpha S_\epsilon$ of order $|\alpha| \leq 2$ with the corresponding unperturbed quantities $\partial^\alpha S$. The comparison in mind is given in terms of estimates exhibiting smallness in terms of the parameter $\epsilon > 0$. If we (for simplicity) assume that V is constant in a neighbourhood of zero then it is straightforward to show that these estimates are uniform not only in $|x| \geq r$ for any $r > 0$ but also in $\lambda \geq 0$. All that is needed to show at this point is, cf. Remark 1.5 2), that

$$\sup_{\lambda\geq 0} \|G_\epsilon - G\|_l \to 0 \text{ as } \|V_\epsilon - V\|_l \to 0. \tag{7.8}$$

For (7.8) we use in turn (and note for comparison) that the estimates of the constants in (1.3) and (1.13) given above are uniform in $\lambda \geq 0$. The constructed solution S_ϵ

has an application in scattering theory at low energies i.e. in the regime $\lambda \to 0$ generalizing parts of [DS], see [Sk].

7.2. **Non-decaying potentials.** Let V be a radial function of class C^l on $\mathbb{R}^d$, $l, d \geq 2$, for which there are constants $a > 0$, $A > 0$, $\mu \in]-\infty, 0]$ and $\sigma \in]0, 2]$ such that

$$-A\langle x\rangle^{-\mu} \leq V(x) \leq -a\langle x\rangle^{-\mu}, \tag{7.9a}$$

$$x \cdot \nabla V(x) + 2V(x) \leq \sigma V(x), \tag{7.9b}$$

$$\partial^\alpha V(x) = O\big(\langle x\rangle^{-(\mu+|\alpha|)}\big) \text{ for } |\alpha| \leq l. \tag{7.9c}$$

Notice that the these conditions are very similar to (7.1a)–(7.1c). The only difference is that now $\mu \leq 0$. We can again allow the inclusion of a positive energy, $\lambda \geq 0$ and obtain uniform estimates. In fact in this case (7.9a)–(7.9c) are invariant under the replacement $V \to V - \lambda$ (up to a change of constants), so if not for the uniformity in $\lambda \geq 0$ we could in the following take $\lambda = 0$. We get the bounds on f in a similar fashion. Note that we need to replace (7.6a) and (7.6b) by

$$\frac{\lambda + a\langle sr\rangle^{-\mu}}{\lambda + A\langle r\rangle^{-\mu}} \geq \frac{a}{A}\frac{\langle sr\rangle^{-\mu}}{\langle r\rangle^{-\mu}}, \tag{7.10a}$$

and

$$\frac{\lambda + A\langle sr\rangle^{-\mu}}{\lambda + a\langle r\rangle^{-\mu}} \leq \frac{\lambda + A\langle r\rangle^{-\mu}}{\lambda + a\langle r\rangle^{-\mu}} \leq \frac{A}{a}, \tag{7.10b}$$

respectively. Again these estimates are for $\lambda \geq 0$, $r \geq 0$ and $s \in [0, 1]$. We take the square root in (7.10a) and (7.10b) and integrate. As for (7.10a) we then use a change of variables and obtain

$$\int_0^1 \sqrt{\frac{\lambda - V(sr(\rho))}{\lambda - V(r(\rho))}}ds \geq \sqrt{\frac{a}{A}}\langle r\rangle^{\mu/2} r^{-1} \int_0^r \langle u\rangle^{-\mu/2} du \geq c.$$

Obviously from (7.10b) we obtain

$$\int_0^1 \sqrt{\frac{\lambda - V(sr(\rho))}{\lambda - V(r(\rho))}}ds \leq \sqrt{\frac{A}{a}} \leq C.$$

We have verified that G given by (7.5a) is a metric of order zero. As before we verify the bound (1.13) with any $\bar{c} \leq \frac{\sigma}{2}/\sup f$, whence $G \in \mathcal{O}$. From this point we can proceed as before and introduce a class of perturbations V_ϵ by (7.7) (now with $\mu \leq 0$) and indeed show the existence of a C^l-solution S_ϵ to the eikonal equation (7.3) with $V \to V_\epsilon$.

We remark that the class of metrics discussed above by perturbing $V(x) = -1/2$ (in the particular case $\mu = \lambda = 0$) coincides with the class considered in [Ba] (here we also take $l = 3$). The parameter $\epsilon > 0$ may play the role of inverse energy. In particular the constructed solution to the eikonal equation was applied in [ACH] in the study of scattering theory of Schrödinger operators with an order zero potential in the high energy regime.

7.3. **Order zero potential, logarithm orbits.** We demonstrate in terms of an example from [HS] (see [HS, Example A.6]) that perturbations of the Euclidean metric in the sense of this paper may involve somewhat exotic geodesics like logarithm orbits. This means that although the direction of any geodesic emanating from 0, say γ, and its velocity $\dot\gamma$ are almost ligned up for all times, cf. Lemma 4.5 ii), the geodesic is permanently rotating around 0 as $|\gamma| \to \infty$.

Consider the symbol h on $\mathbb{R}^2 \times \mathbb{R}^2$ given by $h = h(x,\xi) = g^{-1}\xi^2$, where the conformal (inverse) metric factor is specified in polar coordinates $x = (r\cos\theta, r\sin\theta)$ as $g^{-1} = e^{-2\epsilon f_\epsilon \chi}$; $f_\epsilon = f(\theta - \epsilon \ln r)$, $\chi = \chi(r > 1)$. We assume f is a given smooth 2π–periodic function with $\max f' \geq 1$ and that $\epsilon > 0$ is small. Notice that the "x-space part" of the Hamiltonian orbits of this symbol are the geodesics of the metric $g_{ij} = g\delta_{ij}$. Consider the orbits originating at $(r_0, 0; C, \epsilon C)$ where $C > 0$ is arbitrary and $r_0 > 2$ is determined by the equation

$$f'(\theta_0) = (1+\epsilon^2)^{-1};\ \theta_0 = -\epsilon \ln r_0. \tag{7.11}$$

By assumption there exists at least one such solution. Let us assume that there are only a finite number of such solutions, say θ_j, all being non-degenerate, $f''(\theta_j) \neq 0$. The x-space part of any corresponding orbit is the logarithmic spiral given by the equation $\theta - \epsilon \ln r = \theta_0$. For $f''(\theta_0) > 0$ the orbit corresponds in reduced variables to a saddle, see [HS]. On the other hand for $f''(\theta_0) < 0$ the orbit corresponds to a sink. This means that generically the geodesics of the metric $g_{ij} = g\delta_{ij}$ emanating from 0 are attracted to one of the logarithmic spirals associated to (7.11) and the condition $f''(\theta_0) < 0$.

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