ABSENCE OF EMBEDDED EIGENVALUES FOR RIEMANNIAN LAPLACIANS

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ABSTRACT. In this paper we study absence of embedded eigenvalues for Schrödinger operators on non-compact connected Riemannian manifolds. A principal example is given by a manifold with an end (possibly more than one) in which geodesic coordinates are naturally defined. In this case one of our geometric conditions is a positive lower bound of the second fundamental form of angular submanifolds at infinity inside the end. Another condition is an upper bound of the trace of this quantity, while a third one is a bound of the derivative of part of the trace (some oscillatory behaviour of the trace is allowed). In addition to geometric bounds we need conditions on the potential, a regularity property of the domain of the Schrödinger operator and the unique continuation property. Examples include ends endowed with asymptotic Euclidean or hyperbolic metrics.

CONTENTS

1. INTRODUCTION AND RESULTS

Let (M, g) be a non-compact connected Riemannian manifold of dimension $d \geq 1$ (possibly incomplete), and H the Schrödinger operator on the Hilbert space $\mathcal{H} = L^2(M)$:

$$H = H_0 + V; \quad H_0 = -\tfrac{1}{2}\triangle = \tfrac{1}{2}p_i^* g^{ij} p_j, \quad p_i = -\mathrm{i}\partial_i.$$

We introduce four conditions under which we prove that a self-adjoint realization of H does not have eigenvalues greater than some computable constant. Our conditions appear rather weak and allow for application to manifolds with boundary (possibly caused by metric or potential singularities). In particular, to our knowledge, they are weaker than conditions used so far in the literature on the subject, cf. e.g.

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[Me, MZ, Do, Ku1, Ku2]. The present work is applied in a companion paper [IS] in which scattering theory is studied for a general class of metrics. Our conditions are also weaker than the conditions of [IS].

For the Euclidean case (and a particular subclass of potentials) the theory amounts to absence of positive eigenvalues which is a very well studied subject, see e.g. [RS, FHH2O, JK, KT]. More precisely we recover then [RS, Theorem XIII.58]. On the other hand it does not cover absence of positive eigenvalues for N-body Schrödinger operators, cf. [FH]. The aim of this paper is rather to study absence of embedded (possibly only high energy) eigenvalues of Schrödinger operators in a general geometric framework.

The first condition we impose guarantees intuitively that (M, g) has at least one "expanding end".

Condition 1.1. There exist an unbounded real-valued function $r \in C^\infty(M)$, $r(x) \geq 1$, constants $c_1, c_2 \in \mathbb{R}$ and $c_3 > 0$ with $c_1 > c_2 + c_3$, and a decomposition $\triangle r^2 = \rho_1 + \rho_2$, such that uniformly in $x \in M$ (i.e. the limits in (1.2) and (1.3) are meant to be uniform in $x \in M$):

(1) There exists a constant $r_0 \geq 1$ such that

$$\nabla^2 r^2 \geq (c_1 + \tfrac{1}{2}\rho_1)g \text{ and } \rho_1 \geq -2c_2 \text{ for } r \geq r_0. \tag{1.1}$$

(2) The following bounds hold,

$$\begin{aligned} &\limsup_{r\to\infty} |\mathrm{d}r| < \infty, \quad \limsup_{r\to\infty} \rho_1 < \infty, \\ &\limsup_{r\to\infty} r^{-1}\rho_2 < \infty, \quad \limsup_{r\to\infty} |\mathrm{d}\rho_2| < \infty. \end{aligned} \tag{1.2}$$

(3) Letting $\partial^r = \mathrm{i}p^r = \nabla r = \operatorname{grad} r$ denote the gradient vector field for r, i.e. $\partial^r f = (\partial_i r)g^{ij}(\partial_j f)$ for $f \in C^\infty(M)$, one has

$$\liminf_{r\to\infty} (r\partial^r |\mathrm{d}r|^2 + c_3|\mathrm{d}r|^2) > 0, \quad \lim_{r\to\infty} \partial^r |\mathrm{d}r|^2 = 0. \tag{1.3}$$

Note that the subsets $\{x \in M |\ r(x) \leq \tilde{r}\}$, $\tilde{r} \geq 1$, may not be compact (this is similar to [Ku1, Ku2], see Subsection 2.2). In particular the function r could model a distance function within a fixed single *end* of M extended to be bounded outside, in particular bounded in other ends of M. Note that for an exact distance function the first bound of (1.2) and (1.3) (for any $c_3 > 0$) are trivially fulfilled, and in that case the above operator ∂^r is identified as the geodesic radial derivative ∂_r, see Subsection 2.2. Also note that (1.1) implies that there exists a positive $\tilde{c}_1$ (for example $\tilde{c}_1 = c_1 - c_2$) such that $\nabla^2 r^2 \geq \tilde{c}_1 g > 0$ for $r \geq r_0$. In particular the geodesics in this region are non-trapped, more precisely $r^2 \geq ct^2$ for $t \to \infty$. Another immediate consequence is the lower bound $\liminf_{r\to\infty} r^{-1}\triangle r^2 \geq 0$. Finally it is worth noting that there are no assumptions on derivatives of ρ_1. Motivated by examples, see the discussion before Corollary 2.4, one could think about ρ_1 as a small oscillatory function and the number $\tilde{c}_1$ as a rough bound of its amplitude.

Condition 1.2. There exists a decomposition $V = V_1 + V_2$, $V_1 \in L^2_{\mathrm{loc}}(M)$, $V_2 \in C^1(M)$ and V_1, V_2 real-valued, such that uniformly in $x \in M$:

$$\limsup_{r\to\infty} r|V_1| < \infty, \quad \limsup_{r\to\infty} |V_2| < \infty, \quad \limsup_{r\to\infty} r\partial^r V_2 < \infty. \tag{1.4}$$

Note that under Condition 1.2 the subspace $C_c^\infty(M) \subseteq \mathcal{D}(V)$ and whence that H is defined at least on $C_c^\infty(M)$. However under Conditions 1.1 and 1.2 this operator is not necessarily essentially self-adjoint. Note that (M, g) is allowed to be incomplete and that V is allowed to be unbounded. For instance (M, g) could be the interior of a Riemannian manifold with boundary and for essentially self-adjointness we would then need a symmetric boundary condition. Lack of essential self-adjointness could also originate from unboundedness of V in some end. To fix a self-adjoint extension we first choose a non-negative $\chi \in C^\infty(\mathbb{R})$ with

$$\chi(r) = \begin{cases} 0 & \text{for } r \le 1, \\ 1 & \text{for } r \ge 2, \end{cases}$$

and then set

$$\chi_\nu(r) = \chi(r/\nu), \quad \nu \ge 1. \tag{1.5}$$

We shall henceforth consider the function χ_ν as being composed with the function r from Condition 1.1. In this sense particularly $\chi_\nu \in C^\infty(M)$.

Condition 1.3. The operator H defined on $C_c^\infty(M)$ (by Condition 1.2) has a self-adjoint extension, denoted by H again, such that for any $\psi \in \mathcal{D}(H)$ there exists a sequence $\psi_n \in C_c^\infty(M)$ such that for all large $\nu \ge 1$

$$\|\chi_\nu(\psi - \psi_n)\| + \|\chi_\nu(H\psi - H\psi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Note that Condition 1.3 is fulfilled if (M, g) is complete and V is bounded. In that case indeed H is essentially self-adjoint on $C_c^\infty(M)$, see Proposition 2.1 for a more general result.

As a global condition we impose for this self-adjoint extension *the unique continuation property.*

Condition 1.4. If $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$, $E \in \mathbb{R}$, and $\phi(x) = 0$ in some open subset, then $\phi(x) = 0$ in M.

In Section 2 we shall discuss various models satisfying Conditions 1.1–1.4. We define a "critical" energy,

$$E_0 = \inf_{c \in (0, c_1 - c_2 - c_3]} \limsup_{r \to \infty} \big(V + \tfrac{|\alpha|^2 - c\beta}{c(2c_1 + \rho_1 - 2c)}\big); \tag{1.6a}$$

$$\alpha = \tfrac{1}{4}\mathrm{d}\rho_2 + V_1 \mathrm{d}r^2, \tag{1.6b}$$

$$\beta = (\triangle r^2)V_1 - 2r\partial^r V_2. \tag{1.6c}$$

For some examples in Subsection 2.2 (where for simplicity $V = 0$) we compute that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$, see Examples 2.2 and Remark 2.3 1). Whence for these examples indeed E_0 is critical regarding absence of eigenvalues as stated more generally in the following theorem.

Theorem 1.5. *Suppose Conditions 1.1–1.4. Then the eigenvalues of H are absent above E_0, i.e. $\sigma_{\mathrm{pp}}(H) \cap (E_0, \infty) = \emptyset$.*

Under the above conditions embedded eigenvalues can occur. It is well known in Schrödinger operator theory that the von Neumann Wigner potential, see for example [FH] or [RS, Section XIII.3], provides an example of a positive eigenvalue for a decaying potential $O(r^{-1})$, $r = |x|$. Whence the conclusion of Theorem 1.5 is in general false above the bottom of the essential spectrum. An example of a

Laplace-Beltrami operator having an embedded eigenvalue is constructed in [Ku1]. This is for a hyperbolic metric, and the example shows similarly that the conclusion of Theorem 1.5 in general is false above the bottom of the essential spectrum, see also Remark 2.5 1). (Actually Kumura uses the von Neumann Wigner potential in his construction.)

The proof of Theorem 1.5 follows the scheme of [FHH2O, FH, DeGé, MS] employing in particular a Mourre-type commutator estimate and exponential decay estimates of a priori eigenstates. In our geometric setting the "Mourre commutator" can be very singular (in particular not bounded relatively to H in any usual sense). Consequently we only have a weak (however sufficient) version of the commutator estimate, see Corollary 3.2.

We use throughout the paper the standard notation $\langle\sigma\rangle = (1 + |\sigma|^2)^{1/2}$ and (as above) d for exterior differentiation (acting on functions on M). Note that in local coordinates $p := -\mathrm{id}$ takes the form $p = (p_1, \dots, p_d)$. We shall slightly abuse notation writing for example $p\psi \in \mathcal{H} = L^2(M)$ for $\psi \in C_c^\infty(M)$ even though the correct meaning here is a section of the (complexified) cotangent bundle, i.e. $p\psi \in \Gamma(T^*M)$. Note at this point that $\|p\psi\| := \|p\psi\|_{\Gamma(T^*M)} = \|\,|p\psi|\,\|_{\mathcal{H}}$. If A is an operator on $\mathcal{H}$ and $\psi \in \mathcal{D}(A)$ we denote the expectation $\langle\psi, A\psi\rangle$ by $\langle A\rangle_\psi$. Unimportant positive constants are denoted by C, in particular C may vary from occurrence to occurrence. The dependence on other variables is sometimes indicated by subscripts such as C_ν.

2. Discussion and examples

In this section we investigate how general our conditions are by looking at several examples.

2.1. Global conditions.

We recall some general criteria for self-adjointness and the unique continuation property.

Proposition 2.1. *Let (M, g) be a complete Riemannian manifold of dimension $d \geq 1$. Then the free Schrödinger operator H_0 is essentially self-adjoint on $C_c^\infty(M)$. Suppose V is real-valued, measurable, bounded outside a compact set and in addition: $V \in L^2_{\mathrm{loc}}(M)$ for $d = 1, 2, 3$, $V \in L^p_{\mathrm{loc}}(M)$ for some $p > 2$ if $d = 4$ while $V \in L^{d/2}_{\mathrm{loc}}(M)$ for $d \geq 5$. Then V is relatively compact. In particular H is essentially self-adjoint on $C_c^\infty(M)$.*

We refer to [Ch] and [RS, Theorems X.20 and X.21]. We can generalize the class of potentials to the Stummel class, see e.g. [DoGa].

As for the unique continuation property, Condition 1.4, there is an extensive literature although mostly for Schrödinger operator theory, see e.g. [JK]. For general connected manifolds we refer to [Wo] and references therein, quoting here the following sufficient conditions supplementing connectivity and the conditions in Proposition 2.1: 1) $d = 2, 3, 4$ and V is globally bounded, or 2) $d \geq 5$. One could (of course) add 3) $d = 1$.

2.2. Conditions inside an end.

In the sequel we consider a connected and complete (M, g) of dimension $d \geq 2$ and take (for simplicity) $V = 0$. We shall examine the meaning of Condition 1.1 in the case where, in addition, (M, g) has the following explicit *end* structure: There exists an open subset $E \subset M$ such that isometrically the closure $\bar{E} \cong [0, \infty) \times S$ for some $(d-1)$-dimensional manifold S, and that

$$g = \mathrm{d}r \otimes \mathrm{d}r + g_{\alpha\beta}(r, \sigma)\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta; \quad g_{rr} = 1,\ g_{r\alpha} = g_{\alpha r} = 0, \tag{2.1}$$

where $(r, \sigma) \in [0, \infty) \times S$ denotes local coordinates and the Greek indices run over $2, \ldots, d$. Whence actually r is globally defined in E and it is a smooth distance function (here given as the distance to $\{0\} \times S$). In particular we have $|\mathrm{d}r| = 1$ which obviously implies the first bound of (1.2) and (1.3) with any $c_3 > 0$. Notice here that Condition 1.1 involves only the part of the function r at large values, so in agreement with Condition 1.1 we can cut and extend it to a smooth function on M obeying $r \geq 1$. This is tacitly understood below. To examine the remaining statements (1.1) and (1.2) of Condition 1.1 we compute

$$\nabla^2 r^2 = 2\,\mathrm{d}r \otimes \mathrm{d}r + r(\partial_r g_{\alpha\beta})\,\mathrm{d}\sigma^\alpha \otimes \mathrm{d}\sigma^\beta, \tag{2.2a}$$

$$\triangle r^2 = g^{ij}(\nabla^2 r^2)_{ij} = 2 + r g^{\alpha\beta}(\partial_r g_{\alpha\beta}). \tag{2.2b}$$

2.2.1. *End of warped product type.* If we consider the *warped product* case where $g_{\alpha\beta}(r, \sigma) = f(r) h_{\alpha\beta}(\sigma)$ we obtain, using (2.2a) and (2.2b), the following examples fulfilling also (1.1) and (1.2) of Condition 1.1.

Examples 2.2. (1) Let $f = r^{2a}$ with $a > 0$. Then (1.1) and (1.2) hold with $c_1 = \min\{2, 2a\}$ and $\rho_1 = 0$, and the critical energy $E_0 = 0$.
(2) Let $f = \exp(\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1)$. Then (1.1) and (1.2) hold with $c_1 = 2$ and $\rho_1 = 0$, and $E_0 = 0$.
(3) Let $f = \exp(2\kappa r)$ with $\kappa > 0$. Then (1.1) and (1.2) hold with $c_1 = 2$ and $\rho_1 = 0$, and $E_0 = \kappa^2(d-1)^2/8$.

Remarks 2.3. 1) For all of these examples it is easy to compute that the essential spectrum $\sigma_{\mathrm{ess}}(H_0) \supseteq [E_0, \infty)$. If in addition $M \setminus E$ and S are compact then we have $\sigma_{\mathrm{ess}}(H_0) = [E_0, \infty)$. Whence indeed the absence of eigenvalues in (E_0, ∞) as stated in Theorem 1.5 is optimal under these additional conditions for the above examples (except possibly that the threshold energy $E = E_0$ in a concrete situation might not be an eigenvalue neither).
2) It is not required in Condition 1.1 that r is an exact distance function so we may still have this condition fulfilled in perturbed situations (letting r be the unperturbed distance function). This is also the spirit of [Do, Me, MZ] where (roughly) perturbations of the Euclidean metric (corresponding to $a = 1$ in (1)) are studied. The authors show absence of positive eigenvalues for these models. More generally, but roughly still in the framework of perturbations of (1), absence of embedded eigenvalues was obtained in [Ku2], and for hyperbolic models (roughly for perturbations of (3)) it was done in [Ku1]. However Kumura's results are stated in terms of an exact distance function and all results involve conditions on a radial curvature (possibly including here the radial Ricci curvature). Whence his framework is seemingly somewhat different. It turns out, however, that some of Kumura's conditions appear too strong, in particular curvature conditions are not needed. In Corollary 2.4 below we state a simplified and extended result, see Remark 2.5 2) for further discussion and Corollary 2.6 for an application recovering a main result of [Ku1].

3) Under the condition of warped product metrics growth rates between $f = r^{2a}$ with $a > 1/2$ and $f = \exp(\kappa r^q)$ with $\kappa > 0$ and $q \in (0, 1/2)$ define a class of metrics for which the scattering theory [IS] applies. More generally Conditions 1.1–1.4 are weaker than the conditions used in [IS].

2.2.2. *Volume growth and curvature.* Here we describe the meaning of Condition 1.1 in terms of geometric quantities, and then relate the critical energy E_0 to them. We continue to assume (2.1) in the end E although without warped product structure.

Suppose Condition 1.1 (note that $c_1 - c_2 \leq 2$ is necessary). Then, by (1.1) there exists $\tilde{c}_1 \in [c_1 - c_2, 2]$ such that

$$\nabla^2 r^2 \geq \tilde{c}_1 g \text{ for } r \geq r_0. \tag{2.3}$$

In the coordinates $(r, \sigma) \in [0, \infty) \times S$ used in (2.1) we have (2.2a), so that the inequality (2.3) is equivalent to

$$(r\partial_r g_{\alpha\beta} - \tilde{c}_1 g_{\alpha\beta})_{\alpha,\beta} \geq 0 \text{ for } r \geq r_0. \tag{2.4}$$

Hence, (1.1) implies that the induced metric on the angular manifold $S_{\tilde{r}} = \{x \in \bar{E} |\, r = \tilde{r}\}$ grows as a function of $\tilde{r}$, and $\tilde{c}_1 \in [c_1 - c_2, 2]$ gives a lower growth rate of the metric depending on directions. On the other hand, since we have

$$\triangle r^2 = 2 + 2r\triangle r, \quad \triangle r = \partial_r \ln \sqrt{\det g},$$

and we can measure the volume growth in the radial direction in terms of $\partial_r \ln \sqrt{\det g}$, parts of the assumption (1.2) yield a upper bound for the volume growth. We note that, by taking the trace of (2.4),

$$2r\triangle r \geq \tilde{c}_1(d-1) \text{ for } r \geq r_0,$$

and this implies that the volume has to grow, at least.

Next we assume the "lower metric growth rate" (2.3) for some $\tilde{c}_1 \in (0, 2]$, and "asymptotic volume growth rate"

$$\triangle r = \rho_+ + \tfrac{\omega(r)}{r}; \quad \rho_+ \geq 0, \quad \limsup_{r\to\infty} \omega - \liminf_{r\to\infty} \omega < \tilde{c}_1.$$

Then we shall verify Condition 1.1: By setting

$$\rho_1 = 2\omega, \quad \rho_2 = 2 + 2r\rho_+,$$

and, for sufficiently small $\epsilon > 0$, choosing any c_1, c_2 and c_3 such that

$$0 < \tilde{c}_1 - \limsup_{r\to\infty} \omega - c_1 < \epsilon, \quad 0 < \liminf_{r\to\infty} \omega + c_2 < \epsilon, \quad 0 < c_3 < \epsilon,$$

indeed Condition 1.1 is fulfilled. Hence we can estimate $\inf\{E_0 \,|\, \triangle r^2 = \rho_1 + \rho_2\}$ in terms of the lower metric growth rate and the volume growth rate

$$\begin{aligned} \inf\{E_0 \,|\, \triangle r^2 = \rho_1 + \rho_2\} &\leq \lim_{\epsilon\downarrow 0} \tfrac{1}{2}\rho_+^2/(c_1 - c_2)^2 \\ &= \tfrac{1}{2}\rho_+^2 / \big(\tilde{c}_1 - \limsup_{r\to\infty} \omega + \liminf_{r\to\infty} \omega\big)^2. \end{aligned} \tag{2.5}$$

Note that above $\tilde{c}_1$ is taken as a bound of the amplitude of oscillation allowed in $\triangle r^2$ (i.e. a bound of the term $\rho_1 = 2\omega$ with allowed "bad" derivative). However, also note that in general $\tilde{c}_1$ is just a rough bound because there can be some cancellation in $\nabla^2 r^2 - \frac{1}{2}\rho_1 g$ (an example of this occurs in Corollary 2.4 below).

Now we recover and extend various results of [Ku1, Ku2].

Corollary 2.4. *Suppose* (M, g) *is connected and complete having an end* E *with metric of the form (2.1). Suppose there exist* $\kappa \geq 0$ *and real numbers* $a \leq b$, $a > 0$ *if* $\kappa = 0$, *such that for large* $r \geq 1$

$$(\kappa + \tfrac{a}{r})(g - \mathrm{d}r \otimes \mathrm{d}r) \leq \nabla^2 r_{|S_r} \leq (\kappa + \tfrac{b}{r})(g - \mathrm{d}r \otimes \mathrm{d}r), \tag{2.6}$$

and that

$$A := (d-1)(b-a) < B, \quad B := \begin{cases} \min\{2, a+b\} & \text{if } \kappa = 0, \\ 2 & \text{if } \kappa > 0. \end{cases}$$

Then

$$\sigma_{\mathrm{pp}}(H_0) \cap (\tfrac{\kappa^2}{2}(d-1)^2/(2-A)^2, \infty) = \emptyset. \tag{2.7}$$

Proof. Taking the trace of (2.6), we define ω by

$$\triangle r = \kappa(d-1) + \tfrac{\omega(r)}{r},$$

and set

$$\rho_1 = 2\omega, \quad \rho_2 = 2 + 2r\kappa(d-1).$$

Then, noting for $\kappa = 0$ that there is cancellation of the smallest eigenvalue of $\nabla^2 r|_{S_r}$, we obtain

$$\nabla^2 r^2 - \tfrac{1}{2}\rho_1 g \geq (B - (d-1)b)g, \quad \tfrac{1}{2}\rho_1 \geq (d-1)a.$$

Thus the result follows by applying Theorem 1.5. □

Remarks 2.5.
1) In [Ku1] Kumura constructed an example, fulfilling the conditions of Corollary 2.4 with $\kappa > 0$, for which $\sigma_{\mathrm{ess}}(H_0) = [\frac{\kappa^2}{8}(d-1)^2, \infty)$ and $\sigma_{\mathrm{pp}}(H_0) \cap (\frac{\kappa^2}{8}(d-1)^2, \infty) \neq \emptyset$. Whence in such case (2.7) is an upper bound of the set of embedded eigenvalues. Clearly, as a general feature, the bound is better the smaller $A \geq 0$ can be chosen. In the extreme case, imagining here the quantities a and b being depending on r, where $\liminf a = \limsup b \in \mathbb{R}$ we get an even better bound. We give below an application of Corollary 2.4 to this situation stated in terms of the radial curvature, cf. [Ku1, Theorems 1.4 and 1.7]. Note that the radial curvature can control the second fundamental form $\nabla^2 r_{|S_r}$ by a standard comparison argument, see e.g. [IS, Remark 1.13] for a reference.
2) The bound (1.1) and the parts of the bounds (1.2) may be viewed as bounds on the minimal and the mean curvatures of S_r, respectively, whereas (2.6) certainly is a uniform asymptotic result for all the principal curvatures.

Corollary 2.6. *Suppose* (M, g) *is connected and complete having an end* E *with metric of the form* (2.1). *Suppose there exists* $\kappa > 0$ *such that the radial curvature* R_{rad} *satisfies*

$$R_{\mathrm{rad}} = -\big(\kappa^2 + o(\tfrac{1}{r})\big)g \text{ on } S_r \text{ (uniformly in } x \in E),$$

and there exists $r_1 \geq 0$ *such that*

$$R_{\mathrm{rad}} \leq 0 \text{ on } S_{\tilde{r}} \text{ for all } \tilde{r} \geq r_1 \text{ and } \nabla^2 r \geq 0 \text{ on } S_{r_1}.$$

Then

$$\sigma_{\mathrm{pp}}(H_0) \cap (\kappa^2(d-1)^2/8, \infty) = \emptyset.$$

We note that although the radial curvatures R_{rad} and K_{rad} of [IS] and [Ku1, Ku2], respectively, are different objects they contain equivalent information. Whence in fact the results Corollary 2.6 and [Ku1, Theorem 1.4] (almost) coincide.

3. Mourre-type commutator

Suppose from this point Conditions 1.1–1.4. As a preliminary step in the proof of Theorem 1.5 we show in this section a version of the so-called Mourre estimate. We shall use the Mourre-type commutator with respect to the "conjugate operator"

$$A = \mathrm{i}[H_0, r^2] = \tfrac{1}{2}\{(\partial_i r^2) g^{ij} p_j + p_i^* g^{ij} (\partial_j r^2)\} = r p^r + (p^r)^* r; \quad p^r = -\mathrm{i}\partial^r.$$

While not necessarily being self-adjoint this operator is certainly symmetric as defined on $C_{\mathrm{c}}^\infty(M)$, and that suffices for our applications.

Lemma 3.1. *As a quadratic form on $C_{\mathrm{c}}^\infty(M)$,*

$$\mathrm{i}[H, A] = p_i^* (\nabla^2 r^2 - \tfrac{1}{2}\rho_1 g)^{ij} p_j + \tfrac{1}{2}(\rho_1 H_0 + H_0 \rho_1) + \mathrm{i}\alpha^i p_i - \mathrm{i} p_i^* \alpha^i + \beta,$$

where α and β are defined by (1.6b) and (1.6c), respectively.

Proof. We note the commutator formulas, valid for any $\phi \in C^\infty(M)$,

$$-[H_0, [H_0, \phi]] = p_i^* (\nabla^2 \phi)^{ij} p_j - \tfrac{1}{4}(\triangle^2 \phi), \tag{3.1a}$$

$$p_i^* \phi g^{ij} p_j = \phi H_0 + H_0 \phi + \tfrac{1}{2}(\triangle \phi). \tag{3.1b}$$

As for (3.1a) we refer to [Do, Lemma 2.5] or [IS, Corollary 4.2]. The lemma follows by first using (3.1a) with $\phi = r^2$ and then (3.1b) with $\phi = \frac{1}{2}\rho_1$. □

We introduce for $\sigma \geq 0$

$$H_\sigma = H - \tfrac{\sigma^2}{2} |\mathrm{d}r|^2. \tag{3.2}$$

We shall consider H_σ and as an operator defined on $C_{\mathrm{c}}^\infty(M)$ only. We recall the definitions of χ_ν and E_0, (1.5) and (1.6a), respectively.

Corollary 3.2. *Let $E \in (E_0, \infty)$. There exist $\gamma > 0$ and $C > 0$ such that, if $\nu \geq 1$ is large, then for any $\sigma \geq 0$, as quadratic forms on $C_{\mathrm{c}}^\infty(M)$,*

$$\chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \gamma \chi_\nu^2 - C\chi_\nu (H_\sigma - E)^2 \chi_\nu.$$

Proof. We shall use Lemma 3.1 and in particular the functions α and β appearing there. Choose $\tilde{C}, \gamma > 0$ and $c \in (0, c_1 - c_2 - c_3]$ such that for all large enough $r \geq 1$

$$\nabla^2 r^2 \geq (c_1 + \tfrac{1}{2}\rho_1) g, \tag{3.3a}$$

$$\tilde{C} \geq \tfrac{\rho_1}{2} \geq -c_2, \tag{3.3b}$$

$$r\partial^r |\mathrm{d}r|^2 + c_3 |\mathrm{d}r|^2 \geq 0, \tag{3.3c}$$

$$(2c_1 + \rho_1 - 2c)\big(E - V - \tfrac{|\alpha|^2 - c\beta}{c(2c_1 + \rho_1 - 2c)}\big) \geq 2\gamma. \tag{3.3d}$$

Then by using (3.3a) and the Cauchy Schwarz inequality we obtain for all large $\nu \geq 1$

$$\begin{aligned} \chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \chi_\nu \Big\{ &(c_1 + \tfrac{\rho_1}{2} - c)(H_\sigma - E) + (H_\sigma - E)(c_1 + \tfrac{\rho_1}{2} - c) \\ &+ (2c_1 + \rho_1 - 2c)(E - V + \tfrac{\sigma^2}{2}|\mathrm{d}r|^2) - \tfrac{1}{c}|\alpha|^2 + \beta + \sigma^2 r \partial^r |\mathrm{d}r|^2 \Big\} \chi_\nu. \end{aligned} \tag{3.4}$$

By using in turn (3.3b)–(3.3d) we obtain with $C := (c_1 + \tilde{C} - c)^2/\gamma$

$$\chi_\nu \mathrm{i}[H_\sigma, A]\chi_\nu \geq \chi_\nu \big\{2\gamma - (c_1 + \tfrac{\rho_1}{2} - c)^2/C - C(H_\sigma - E)^2\big\}\chi_\nu$$

and whence the assertion. □

4. Exponential decay of eigenstates

The proof of Theorem 1.5, given in this section, depends on the following exponential decay estimate which in turn will be proved in Section 5.

Proposition 4.1. *Let $E \in \sigma_{\mathrm{pp}}(H) \cap (E_0, \infty)$ and suppose $\phi \in \mathcal{D}(H)$ satisfies $H\phi = E\phi$. Then for any $\sigma \geq 0$ one has $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$.*

To implement Condition 1.3 efficiently we need to strengthen the stated approximation property under some additional conditions (fulfilled for eigenstates due to Proposition 4.1).

Lemma 4.2. *Let $\psi \in \mathcal{D}(H)$. There exists $\nu_0 \geq 1$ such that for $\nu \geq \nu_0$ and for any $\sigma \geq 0$ such that $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$ the following properties hold: The states $\chi_\nu \mathrm{e}^{\sigma r}p\psi, \mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$ and there exists a sequence $\psi_n \in C_{\mathrm{c}}^\infty(M)$ (possibly depending on σ) such that as $n \to \infty$*

$$\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_n)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(p\psi - p\psi_n)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_n)\| \to 0. \tag{4.1}$$

Proof. Step I Note the distributional identity

$$\chi_\nu \mathrm{e}^{\sigma r}p\psi = \mathrm{e}^{\sigma r}p\chi_\nu\psi + \mathrm{i}\mathrm{e}^{\sigma r}\psi\chi_\nu' \mathrm{d}r.$$

Applied to the given ψ we see that $\chi_\nu \mathrm{e}^{\sigma r}p\psi \in \mathcal{H}$ if and only if $\mathrm{e}^{\sigma r}p\chi_\nu\psi \in \mathcal{H}$.
Step II We claim that there exists $C > 0$ such that, if $\nu \geq 1$ is large, then for any $\psi \in C_{\mathrm{c}}^\infty(M)$ and $\sigma \geq 0$

$$\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\|^2 \leq \|\chi_\nu \mathrm{e}^{\sigma r}H\psi\|^2 + C\langle\sigma\rangle^2\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\|^2. \tag{4.2}$$

In fact by (3.1b)

$$\begin{aligned}\|\chi_\nu \mathrm{e}^{\sigma r}|p\psi|\|^2 &= 2\,\mathrm{Re}\,\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, \chi_\nu \mathrm{e}^{\sigma r}H\psi\rangle + \tfrac{1}{2}\langle\psi, (\triangle\chi_\nu^2 \mathrm{e}^{2\sigma r})\psi\rangle - 2\langle\chi_\nu \mathrm{e}^{\sigma r}\psi, V\chi_\nu \mathrm{e}^{\sigma r}\psi\rangle \\ &\leq \|\chi_\nu \mathrm{e}^{\sigma r}H\psi\|^2 + C\langle\sigma\rangle^2\|\chi_{\nu/2}\mathrm{e}^{\sigma r}\psi\|^2.\end{aligned}$$

Here we used Condition 1.1 and the following consequence

$$|\triangle r| = \tfrac{1}{2r}|(\triangle r^2) - 2|\mathrm{d}r|^2| \leq C \text{ for } r = r(x) \text{ large}. \tag{4.3}$$

Step III We consider the case $\sigma = 0$, and hence suppose only $\psi \in \mathcal{D}(H)$. Let $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geq 1$ be as in Condition 1.3. Then, regarding (4.1), it suffices to consider the middle term. By (4.2) we have

$$\|\chi_\nu(p\psi_n - p\psi_{n'})\|^2 \leq C\big(\|\chi_\nu(H\psi_n - H\psi_{n'})\|^2 + \|\chi_{\nu/2}(\psi_n - \psi_{n'})\|^2\big).$$

This implies $\chi_\nu p\psi_n$ converges strongly. Since also $\chi_\nu p\psi_n$ converges in distributional sense to $\chi_\nu p\psi$, we obtain that the limit $\chi_\nu p\psi \in \mathcal{H}$ and then in turn, by letting $n' \to \infty$ above, (4.1) for $\sigma = 0$.
Step IV We let $\sigma > 0$ and suppose $\mathrm{e}^{\sigma r}\psi, \mathrm{e}^{\sigma r}H\psi \in \mathcal{H}$. Choose $\psi_n \in C_{\mathrm{c}}^\infty(M)$ and large $\nu \geq 1$ as in Condition 1.3, again. As for the first and the third terms of (4.1), we compute as follows: Put $\psi_{n,\nu'} = \bar{\chi}_{\nu'}\psi_n$ for $\nu' \geq 2\nu$ and with $\bar{\chi}_{\nu'} := 1 - \chi_{\nu'}$. Then we decompose

$$\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_{n,\nu'}) = \bar{\chi}_{\nu'}\mathrm{e}^{\sigma r}\chi_\nu(\psi - \psi_n) + \chi_{\nu'}\mathrm{e}^{\sigma r}\psi. \tag{4.4}$$

We put

$$R_{\nu'} = \mathrm{i}[H, \chi_{\nu'}] = \tfrac{1}{2}(\chi'_{\nu'} p^r + (p^r)^* \chi'_{\nu'}) = \chi'_{\nu'} p^r - \tfrac{\mathrm{i}}{2}\big(\chi''_{\nu'} |\mathrm{d}r|^2 + \chi'_{\nu'} \triangle r\big), \tag{4.5}$$

and decompose similarly

$$\begin{aligned} &\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_{n,\nu'}) \\ &= \bar{\chi}_{\nu'} \mathrm{e}^{\sigma r} \chi_\nu (H\psi - H\psi_n) + \chi_{\nu'} \mathrm{e}^{\sigma r} H\psi + \mathrm{i}\mathrm{e}^{\sigma r} R_{\nu'}(\psi - \psi_n) - \mathrm{i}\mathrm{e}^{\sigma r} R_{\nu'} \psi. \end{aligned} \tag{4.6}$$

The norm of the right-hand side of (4.4) can be arbitrarily small by first letting ν' be large and then n large accordingly (using that $\bar{\chi}_{\nu'} \mathrm{e}^{\sigma r}$ is bounded). Similarly the norm of first three terms on the right-hand side of (4.6) can be arbitrarily small by first letting ν' be large and then n large accordingly (for the third term we use Step III, i.e. (4.1) with $\sigma = 0$). It remains to consider the last term on the right-hand side of (4.6). We claim that

$$\|\mathrm{e}^{\sigma r} R_{\nu'} \psi\| \leq C/\nu'. \tag{4.7}$$

To show this we use again Step III to write

$$\|\chi'_{\nu'} \mathrm{e}^{\sigma r} p\psi\|^2 = \lim_{m\to\infty} \|\chi'_{\nu'} \mathrm{e}^{\sigma r} p\psi_m\|^2.$$

On the other hand by the derivation of (4.2)

$$\|\chi'_{\nu'} \mathrm{e}^{\sigma r} p\psi_m\|^2 \leq C\big(\|\chi'_{\nu'} \mathrm{e}^{\sigma r} H\psi_m\|^2 + \big(\tfrac{\langle\sigma\rangle}{\nu'}\big)^2 \|\chi_{\nu/2} \bar{\chi}_{2\nu'} \mathrm{e}^{\sigma r} \psi_m\|^2\big),$$

and hence we conclude by taking the limit that

$$\begin{aligned} \|\chi'_{\nu'} \mathrm{e}^{\sigma r} p\psi\|^2 &\leq \big(\tfrac{C_\sigma}{\nu'}\big)^2 \big(\|\chi_\nu \bar{\chi}_{2\nu'} \mathrm{e}^{\sigma r} H\psi\|^2 + \|\chi_{\nu/2} \bar{\chi}_{2\nu'} \mathrm{e}^{\sigma r} \psi\|^2\big) \\ &\leq \big(\tfrac{C_\sigma}{\nu'}\big)^2 \big(\|\mathrm{e}^{\sigma r} H\psi\|^2 + \|\mathrm{e}^{\sigma r} \psi\|^2\big). \end{aligned} \tag{4.8}$$

A consequence of (4.8) is indeed (4.7), and whence in turn also the last term on the right-hand side of (4.6) is small for ν' sufficiently large.

We conclude that there exists a sequence of indices $(\nu'(m), n(m))$ so that with $\psi_m := \psi_{n(m),\nu'(m)}$ (here and henceforth slightly abusing notation)

$$\|\chi_\nu \mathrm{e}^{\sigma r}(\psi - \psi_m)\| + \|\chi_\nu \mathrm{e}^{\sigma r}(H\psi - H\psi_m)\| \to 0.$$

In particular, using here (4.2), the right-hand side of

$$\|\chi_{2\nu} \mathrm{e}^{\sigma r} p(\psi_n - \psi_{n'})\|^2 \leq C\big(\|\chi_{2\nu} \mathrm{e}^{\sigma r} H(\psi_n - \psi_{n'})\|^2 + \|\chi_\nu \mathrm{e}^{\sigma r}(\psi_n - \psi_{n'})\|^2\big)$$

is small for $n, n' \to \infty$. We can from this point mimic the last part of Step III. □

Proof of Theorem 1.5. Suppose $E \in \sigma_{\mathrm{pp}}(H) \cap (E_0, \infty)$ and let ϕ be any corresponding eigenstate. Then, by Proposition 4.1, for any $\nu \geq 1$ and $\sigma \geq 0$

$$\phi_\sigma = \phi_{\sigma,\nu} := \chi_\nu \mathrm{e}^{\sigma(r - 4\nu)} \phi \in \mathcal{H}. \tag{4.9}$$

We will choose $\nu \geq 1$ large in agreement with Lemma 4.2 with $\psi = \phi$. In the following computations we actually have to first choose an approximate sequence for ϕ from $C_{\mathrm{c}}^\infty(M)$ and then take the limits. This can be done by using Lemma 4.2 and the closedness of H, but since the verification is rather straightforward we shall not elaborate on this point.

Put $H_\sigma = H - \frac{\sigma^2}{2}|\mathrm{d}r|^2$ as in (3.2) and fix $c \in (0, c_1 - c_2 - c_3]$ as in proof of Corollary 3.2, then by (3.4) for large $\nu \geq 1$

$$\sigma^2 \langle r\partial^r |\mathrm{d}r|^2 + c_3 |\mathrm{d}r|^2 \rangle_{\phi_\sigma} \leq \langle \mathrm{i}[H_\sigma, A] \rangle_{\phi_\sigma} - \mathrm{Re}\,\langle (2c_1 + \rho_1 - 2c)(H_\sigma - E) \rangle_{\phi_\sigma}. \tag{4.10}$$

We compute the terms on the right-hand side. We note, putting $R_\nu = \mathrm{i}[H_0, \chi_\nu] = \mathrm{Re}\,\big(\chi_\nu' p^r\big)$ as in (4.5),

$$(H_\sigma - E)\phi_\sigma = -\,\mathrm{i}\sigma(\mathrm{Re}\,p^r)\phi_\sigma - \mathrm{i}\mathrm{e}^{\sigma(r-4\nu)}R_\nu\phi. \tag{4.11}$$

In particular indeed $\phi_\sigma \in \mathcal{D}(H)$. Hence we can write (4.10) as

$$\begin{aligned}
&\langle \mathrm{i}[H_\sigma, A]\rangle_{\phi_\sigma} - \mathrm{Re}\,\langle (2c_1 + \rho_1 - 2c)(H_\sigma - E)\rangle_{\phi_\sigma} \\
&= -2\sigma\,\mathrm{Re}\,\langle (\mathrm{Re}\,p^r)A\rangle_{\phi_\sigma} - 2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi, \\
&\quad - \sigma\,\mathrm{Im}\,\langle (2c_1 + \rho_1 - 2c)\,\mathrm{Re}\,p^r\rangle_{\phi_\sigma} - \mathrm{Im}\,\langle (2c_1 + \rho_1 - 2c)\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)}R_\nu\rangle_\phi.
\end{aligned} \tag{4.12}$$

The first and third terms of (4.12) are estimated using, for large $r \geq 1$,

$$\begin{aligned}
&-2\,\mathrm{Re}\,((\mathrm{Re}\,p^r)A) - \mathrm{Im}\,(2c_1 + \rho_1 - 2c)\,\mathrm{Re}\,p^r \\
&= -(\mathrm{Re}\,p^r)(2r\,\mathrm{Re}\,p^r - \mathrm{i}|\mathrm{d}r|^2) - \tfrac{1}{2\mathrm{i}}(2c_1 + \rho_1 - 2c)\,\mathrm{Re}\,p^r + \mathrm{h.c.} \\
&\leq -(\mathrm{Re}\,p^r)(2r - 1)\,\mathrm{Re}\,p^r + C + (\partial^r|\mathrm{d}r|^2) \\
&\leq C + 1.
\end{aligned}$$

As for the second term of (4.12) we estimate (recall the notation $\bar\chi_\nu = 1 - \chi_\nu$)

$$\begin{aligned}
&-2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi \\
&\leq \|\mathrm{e}^{\sigma(r-4\nu)}R_\nu\phi\|^2 + \|\bar\chi_{2\nu}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\phi\|^2 \\
&\leq \big\{\|\chi_\nu' \mathrm{e}^{\sigma(r-4\nu)}p^r\phi\| + \tfrac12\|(\chi_\nu''|\mathrm{d}r|^2 + \chi_\nu'(\triangle r))\mathrm{e}^{\sigma(r-4\nu)}\phi\|\big\}^2 \\
&\quad + \big\{\|2r\bar\chi_{2\nu}\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}p^r\phi\| + \|\bar\chi_{2\nu}(2r|\mathrm{d}r|^2\chi_\nu' + 2\sigma r\chi_\nu|\mathrm{d}r|^2 + \tfrac12(\triangle r^2)\chi_\nu)\mathrm{e}^{\sigma(r-4\nu)}\phi\|\big\}^2 \\
&\leq C\nu^2\|\chi_{\nu/2}|p\phi|\|^2 + C\nu^2\langle\sigma\rangle^2\|\phi\|^2,
\end{aligned}$$

where we have used (4.3). Note that $C > 0$ does not depend on ν or σ because $r \leq 2\nu$ on $\mathrm{supp}\,\chi_\nu'$. By using (4.1) and (4.2) (both with $\sigma = 0$) we then conclude

$$-2\,\mathrm{Re}\,\langle R_\nu \mathrm{e}^{\sigma(r-4\nu)}A\chi_\nu \mathrm{e}^{\sigma(r-4\nu)}\rangle_\phi \leq C\nu^2\langle\sigma\rangle^2\|\phi\|^2.$$

Finally, we examine the fourth term of (4.12). Note that we can not differentiate ρ_1. But by the support property of χ_ν' (the one used before) the term is estimated similarly to the second term of (4.12), and we obtain

$$-\,\mathrm{Im}\,\langle (2c_1 + \rho_1 - 2c)\chi_\nu \mathrm{e}^{2\sigma(r-4\nu)}R_\nu\rangle_\phi \leq C\|\phi\|^2.$$

We summarize

$$\sigma^2\langle r(\partial^r|\mathrm{d}r|^2) + c_3|\mathrm{d}r|^2\rangle_{\phi_\sigma} - C\sigma\|\phi_\sigma\|^2 \leq C\nu^2\langle\sigma\rangle^2\|\phi\|^2. \tag{4.13}$$

We shall apply (4.13) to a fixed $\nu \geq 1$ chosen so large that the quantity $r(\partial^r|\mathrm{d}r|^2) + c_3|\mathrm{d}r|^2$ is greater than some positive constant on $\mathrm{supp}\,\chi_\nu$.

Now assume $\chi_{5\nu}\phi \not\equiv 0$. After division by $\langle\sigma\rangle^2$ on both sides of (4.13) the left-hand side grows exponentially as $\sigma \to \infty$ whereas the right-hand side is bounded, and hence we obtain a contradiction. Thus $\chi_{5\nu}\phi \equiv 0$, and then by Condition 1.4 we conclude that $\phi(x) = 0$ in M. □

5. Auxiliary operators

In this section we give the proof of Proposition 4.1. We introduce regularized weights

$$\theta_m(r) = r(1+\tfrac{r}{m})^{-1}, \quad m \geq 1,$$

and denote the derivatives in r by $\theta_m^{(k)}(r)$, e.g.,

$$\theta_m'(r) = \theta_m^{(1)}(r) = (1+\tfrac{r}{m})^{-2}.$$

We introduce furthermore

$$\Theta_m(r) = \Theta_m^{\sigma,\delta}(r) = \sigma r + \delta\theta_m(r), \quad \sigma,\delta \geq 0,$$

and denote the derivatives by $\Theta_m^{(k)}(r)$ as above. Now we define some observables:

$$B = \mathrm{i}[H_0, r] = \tfrac{1}{2}(p^r + (p^r)^*) = p^r + \tfrac{1}{2\mathrm{i}}(\triangle r),$$
$$B_m = \mathrm{i}[H_0, \Theta_m] = \tfrac{1}{2}\left(\Theta_m' p^r + (p^r)^*\Theta_m'\right) = \Theta_m' p^r + \tfrac{1}{2\mathrm{i}}\{(\triangle r)\Theta_m' + |\mathrm{d}r|^2\Theta_m''\},$$
$$R_\nu = \mathrm{i}[H_0, \chi_\nu] = \tfrac{1}{2}(\chi_\nu' p^r + (p^r)^*\chi_\nu'), \quad \nu \geq 1.$$

Then we have the properties:

$$A = 2Br - \tfrac{1}{\mathrm{i}}|\mathrm{d}r|^2 = 2rB + \tfrac{1}{\mathrm{i}}|\mathrm{d}r|^2 \tag{5.1a}$$
$$B_m = B\Theta_m' - \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2\Theta_m'' = \Theta_m' B + \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2\Theta_m'', \tag{5.1b}$$
$$\begin{aligned}(B_m)^2 &= B(\Theta_m')^2B - \tfrac{1}{2}(\partial^r|\mathrm{d}r|^2)\Theta_m'\Theta_m'' - \tfrac{1}{2}|\mathrm{d}r|^4\Theta_m'\Theta_m''' - \tfrac{1}{4}|\mathrm{d}r|^4(\Theta_m'')^2\\ &\leq B(\Theta_m')^2B + C\delta(\sigma+\delta),\end{aligned} \tag{5.1c}$$

where the last inequality is for large r. We set for $\nu' \geq 2\nu$ and $\psi \in C_{\mathrm{c}}^\infty(M)$

$$\psi_m = \psi_{m,\nu,\nu'} = \chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}\psi; \quad \chi_{\nu,\nu'} = \chi_\nu\bar\chi_{\nu'}, \quad \bar\chi_{\nu'} = 1 - \chi_{\nu'},$$

not to be mixed up with ψ_n in Lemma 4.2. We recall the notation (3.2). A computation shows, cf. (4.11), that

$$\begin{aligned}&\mathrm{i}(H_\sigma - E)\psi_m\\ &= \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi + \left\{B_m - \tfrac{1}{2\mathrm{i}}((\Theta_m')^2 - \sigma^2)|\mathrm{d}r|^2\right\}\psi_m + \mathrm{e}^{\Theta_m}(R_\nu - R_{\nu'})\psi.\end{aligned} \tag{5.2}$$

Lemma 5.1. *Let $\sigma_0 \geq 0$ be fixed.*

(i) *Let $\epsilon > 0$. Then there exists $C > 0$ such that, if $\nu \geq 1$ is large, for any $m \geq 1$, $0 \leq \delta \leq 1$ and $0 \leq \sigma \leq \sigma_0$, as quadratic forms on $C_{\mathrm{c}}^\infty(M)$,*

$$\chi_\nu \operatorname{Re}(AB_m)\chi_\nu \geq 2\chi_\nu Br\Theta_m' B\chi_\nu - (\epsilon + C\delta)\chi_\nu^2.$$

(ii) *Let $\epsilon' > 0$. Then there exists $C > 0$ such that, if $\nu \geq 1$ is large, for any $\nu' \geq 2\nu$, $m \geq 1$, $0 \leq \delta \leq 1$, $0 \leq \sigma \leq \sigma_0$, $E \in \mathbb{R}$ and $\psi \in C_{\mathrm{c}}^\infty(M)$*

$$\begin{aligned}&\|(H_\sigma - E)\psi_m\|^2\\ &\leq 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + \epsilon'\langle Br\Theta_m' B\rangle_{\psi_m} + C\delta\|\psi_m\|^2\\ &\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C(\nu')^{-2}(\|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}\psi\|^2 + \|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}p\psi\|^2).\end{aligned}$$

Proof. (i) By (5.1a) and (5.1b)

$$\begin{aligned}\operatorname{Re}(AB_m) &= \tfrac{1}{2}(2Br - \tfrac{1}{\mathrm{i}}|\mathrm{d}r|^2)(\Theta_m' B + \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2\Theta_m'') + \text{h.c.}\\ &= Br\Theta_m' B + \tfrac{1}{2\mathrm{i}}Br|\mathrm{d}r|^2\Theta_m'' - \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2\Theta_m' B + \tfrac{1}{4}|\mathrm{d}r|^4\Theta_m'' + \text{h.c.}\\ &= 2Br\Theta_m' B - \tfrac{1}{2}\left\{(\partial^r|\mathrm{d}r|^2)(\Theta_m' + r\Theta_m'') + |\mathrm{d}r|^4(\Theta_m'' + r\Theta_m''')\right\}.\end{aligned}$$

Then by the first bound of (1.2) and (1.3) the assertion follows.

(ii) By (5.2), (5.1c), the first bound of (1.2) and (4.3)

$$\begin{aligned}
&\|(H_\sigma - E)\psi_m\|^2 \\
&\le 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + 5\langle (B_m)^2\rangle_{\psi_m} + \tfrac{5}{4}\|((\Theta_m')^2-\sigma^2)|\mathrm{d}r|^2\psi_m\|^2 \\
&\quad + 5\|\mathrm{e}^{\Theta_m}R_\nu\psi\|^2 + 5\|\mathrm{e}^{\Theta_m}R_{\nu'}\psi\|^2 \\
&\le 5\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + 5\langle B(\Theta_m')^2B\rangle_{\psi_m} + C\delta\|\psi_m\|^2 \\
&\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C(\nu')^{-2}(\|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}\psi\|^2 + \|\chi_{\nu,2\nu'}\mathrm{e}^{\Theta_m}p\psi\|^2).
\end{aligned}$$

Now choose $\nu \ge 1$ large enough so that $5\Theta_m' \le 5(\sigma_0+1) \le \epsilon' r$ on $\operatorname{supp}\chi_\nu$, and we are done. □

Proof of Proposition 4.1. We let E and ϕ be as in the proposition. Set

$$\sigma_0 = \sup\{\sigma \ge 0 |\ \mathrm{e}^{\sigma r}\phi \in \mathcal{H}\},$$

and assume $\sigma_0 < \infty$. If $\sigma_0 > 0$ we choose $\sigma \in [0,\sigma_0)$ and a small $\delta > 0$ such that $\sigma + \delta > \sigma_0$. If $\sigma_0 = 0$ we set $\sigma = 0$ and choose a small $\delta > 0$. These numbers will be determined more precisely in the following arguments. In any case we have $\mathrm{e}^{\sigma r}\phi \in \mathcal{H}$. We indicate below the dependence of constants using subscripts.

Due to Corollary 3.2, for any $\psi \in C_{\mathrm{c}}^\infty(M)$

$$\|\psi_m\|^2 \le \gamma^{-1}\langle \mathrm{i}[H_\sigma, A]\rangle_{\psi_m} + C_0\|(H_\sigma - E)\psi_m\|^2;\ C_0 = C/\gamma. \tag{5.3}$$

We estimate the right-hand side using Lemma 5.1. For the first term of (5.3) we use (5.2) and Lemma 5.1(i) with $\epsilon = \frac{\gamma}{3}$ estimating

$$\begin{aligned}
&\langle \mathrm{i}[H_\sigma, A]\rangle_{\psi_m} \\
&= -\langle \mathrm{i}(H_\sigma - E)\psi_m, A\psi_m\rangle + \text{h.c.} \\
&= -\langle \mathrm{i}\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi, A\psi_m\rangle - \langle B_m\psi_m, A\psi_m\rangle + \langle \tfrac{1}{2\mathrm{i}}|\mathrm{d}r|^2((\Theta_m')^2-\sigma^2)\psi_m, A\psi_m\rangle \\
&\quad - \langle \mathrm{e}^{\Theta_m}(R_\nu - R_{\nu'})\psi, A\psi_m\rangle + \text{h.c.} \\
&\le 2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|\|A\psi_m\| - 2\operatorname{Re}\langle AB_m\rangle_{\psi_m} - \langle (r\partial^r|\mathrm{d}r|^2)((\Theta_m')^2-\sigma^2)\rangle_{\psi_m} \\
&\quad - \langle 2r|\mathrm{d}r|^4\Theta_m'\Theta_m''\rangle_{\psi_m} + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) \\
&\quad + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2) \\
&\le C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 - 4\langle Br\Theta_m'B\rangle_{\psi_m} + (\tfrac{2\gamma}{3} + C_1\delta)\|\psi_m\|^2 \\
&\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2),
\end{aligned}$$

where we used that $r/\nu' \le 2\sqrt{r/\nu'}$ on $\operatorname{supp}\chi_{\nu,2\nu'}$ to estimate $(\nu')^{-2}\|A\psi_m\|^2$.

On the other hand, for the second term of (5.3), let us choose $\epsilon' = \frac{4}{\gamma C_0}$ in Lemma 5.1(ii). Then (5.3) is estimated as

$$\begin{aligned}
\|\psi_m\|^2 &\le C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + \Big(\tfrac{2}{3} + (\tfrac{C_1}{\gamma} + C_2)\delta\Big)\|\psi_m\|^2 \\
&\quad + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2) + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2).
\end{aligned}$$

Now fix $\nu \geq 1$ sufficiently large (so that the above estimates hold), and let σ and δ be such that $\frac{2}{3} + (\frac{C_1}{\gamma} + C_2)\delta \leq \frac{3}{4}$ and $\sigma + \delta > \sigma_0$. Then

$$\begin{aligned}\tfrac{1}{4}\|\psi_m\|^2 \leq C(\nu')^2\|\chi_{\nu,\nu'}\mathrm{e}^{\Theta_m}(H-E)\psi\|^2 + C_\nu(\|\chi_{\nu/2}\psi\|^2 + \|\chi_{\nu/2}p\psi\|^2)\\ + C_m(\|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}\psi\|^2 + \|\sqrt{r/\nu'}\chi_{\nu,2\nu'}\mathrm{e}^{\sigma r}p\psi\|^2).\end{aligned} \tag{5.4}$$

By Lemma 4.2 we can replace ψ of (5.4) by ϕ. This makes the first term on the right-hand side disappear. Next let $\nu' \to \infty$ invoking Lebesgue's dominated convergence theorem. Note that the third term disappears, and consequently we are left with the bound

$$\|\chi_\nu \mathrm{e}^{\Theta_m}\phi\|^2 \leq 4C_\nu(\|\chi_{\nu/2}\phi\|^2 + \|\chi_{\nu/2}p\phi\|^2). \tag{5.5}$$

By letting $m \to \infty$ in (5.5) invoking Lebesgue's monotone convergence theorem we conclude that $\chi_\nu \mathrm{e}^{(\sigma+\delta)r}\phi \in \mathcal{H}$. This is a contradiction since $\sigma + \delta > \sigma_0$. □

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