

**CLASSICAL AND QUANTUM DYNAMICS FOR
2D-ELECTROMAGNETIC POTENTIALS ASYMPTOTICALLY
HOMOGENEOUS OF DEGREE ZERO**

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ABSTRACT. We consider a charged particle moving in the plane subject to electromagnetic potentials with non-vanishing radial limits. We analyse the classical and the quantum dynamics for large time in the case the angular part of the (limiting) Lorentz force (defined for velocities that are purely radial) has a finite number of zeros at fixed energy. Any such zero defines a channel, and to the “stable” ones we associate quantum wave operators. Their completeness is studied in the case of zero as well as nonzero magnetic flux. In the latter case one needs possibly to incorporate a channel of spiraling states. These states are similar to those studied recently in the sign-definite case in [CHS].

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1. INTRODUCTION AND RESULTS

In this paper we continue the study initiated in [CHS] of the dynamics of a charge 1 particle moving in the plane subject to a very long-range magnetic field \mathbf{B} perpendicular to the plane. More precisely we assume $\mathbf{B} = (0, 0, r^{-1}b(x))$ where x is the position in the plane and $r = |x|$, and that $b(\theta) := \lim_{r \rightarrow \infty} b(x)$ is a non-vanishing smooth function on the unit-circle.

The classical Hamiltonian may be put in the form

$$h = h(x, \xi) = 2^{-1}(\xi - \mathbf{a})^2, \quad (1.1)$$

with $\mathbf{a}(x) = \int_0^1 r^{-1}b(sx)(-x_2, x_1)ds$ where $x = (x_1, x_2)$. Here the mass is taken to be 1.

In the bulk of this paper we shall consider the more general symbol

$$h = h(x, \xi) = 2^{-1}(\xi - \mathbf{a})^2 + V, \quad (1.2)$$

where $V(\theta) = \lim_{r \rightarrow \infty} V(x)$ similarly defines a smooth function on the unit-circle. From this point of view the present paper may be seen as a natural continuation/generalization (in the 2-dimensional framework) of the previous works [CHS], [He], [HMV], [HS1] and [HS2].

To simplify the presentation let us assume throughout this introduction that $V = 0$, and that (using polar coordinates) $\mathbf{a}(x) = \mathbf{a}(\theta) = (-\sin \theta, \cos \theta)b(\theta)$. In the context of quantum mechanics discussed below we take $\mathbf{a}(x)$ to be regularized at the (singular) origin, but keep the above form outside a compact set. Let us denote by H the corresponding quantization which is a self-adjoint operator on $L^2(\mathbb{R}_x^2)$. It is a general fact that the spectrum is given by $\sigma(H) = \sigma_{ess}(H) = [0, \infty)$; see [CFKS, Theorem 6.1].

In [CHS] we studied the case where $b(\theta) < 0$ for all angles. In outline we showed that above a certain energy $E_d > 0$ all classical scattering orbits will go to infinity along logarithmic spirals, and an analogous result was proved for scattering quantum states localized in (E_d, ∞) . In this energy regime the singular continuous spectrum and the pure point spectrum are empty and discrete, respectively. At the particular energy E_d there exists a one-parameter family of closed classical orbits. Below E_d the classical orbits have infinitely many ‘‘loops’’. The nature of the spectrum of H in this regime is not known except in the constant b case where it is pure point, cf. [CFKS, Theorem 6.2].

Now we shall study the case where $b(\theta)$ has a finite number of non-degenerate zeros. We group those zeros into the ‘‘stable’’ ones where $b'(\theta_j) > 0$ and the ‘‘unstable’’ ones where $b'(\theta_j) < 0$. In either case there is a channel associated to θ_j . More precisely there are classical scattering orbits with $\theta(t) \rightarrow \theta_j$ as $t \rightarrow \infty$. (Here and henceforth $t \rightarrow \infty$ means $t \rightarrow +\infty$.) However they are of a different nature

in the following sense: There is a reduced phase space in which the stable θ_j 's correspond to stable fixed points, while the unstable θ_j 's correspond to unstable fixed points (motivating their names). This means that the orbits associated to an unstable θ_j are “rare” (by the stable manifold theorem in the theory of dynamical systems), while the orbits associated with a stable θ_j fill out a continuum. The unstable channels do not exist in quantum mechanics.

For other works on quantum scattering theory for magnetic Hamiltonians we refer to [Hö2], [LT1], [LT2], [E], [NR], [R] and [RY]. All these works require decay of the magnetic vector potential (see [CHS] for a more detailed account). In the case of periodic fields it was shown in [BS] and [S] that the spectrum of the Hamiltonian is purely absolutely continuous. There are other works on $2D$ -magnetic Hamiltonians with continuous spectrum, [Iw], [MP], [BP] and [FGW]. Those do in fact deal with more “long-range” vector potentials than considered in this paper (although strictly disjoint from our class). The books [CFKS], [DG2] and [RS] contain further background information.

1.1. Results in classical mechanics. For simplicity we take henceforth $V = 0$. We distinguish between the two cases 1) the flux $\int_0^{2\pi} b(\theta)d\theta = 0$, and 2) the flux $\int_0^{2\pi} b(\theta)d\theta < 0$. Partly motivated by the fact that there are collapsing orbits, $r(t) = |x(t)| \rightarrow 0$ in finite time, let us define a “classical scattering orbit” to be a solution to Hamilton’s (or Newton’s) equations with $r(t) \rightarrow \infty$ for $t \rightarrow \infty$.

1.1.1. Zero flux. In this case *any* scattering orbit approaches one of the zeros, $b(\theta_j) = 0$, meaning $\theta(t) \rightarrow \theta_j$. Moreover there exists the limit $\lim_{t \rightarrow \infty} t^{-1}r(t) = \sqrt{2E}$, where $E > 0$ denotes the energy of the orbit.

1.1.2. Negative flux. We are interested in 2π -periodic solutions $\rho = \rho_E$ to the system

$$\begin{cases} \frac{d\rho}{d\theta} = b + \eta \\ \eta = \sqrt{2E - \rho^2} > 0, \int_0^{2\pi} \frac{\rho}{\eta} d\theta > 0 \end{cases} \quad (1.3)$$

Here ρ and η represent the radial and angular part of the velocity, respectively. We know from [CHS] that if $b < 0$, then there exists exactly one such solution for all high enough energies and it defines a logarithmic spiral.

We show in the present case where b has zeros:

- i) There is at most one positive 2π -periodic solution ρ_E to (1.3) for fixed E . If the set \mathcal{E} of energies where the solution exists is non-empty, then it is an open bounded interval (E_d, E_e) .
- ii) If ρ_E exists, then for *any* scattering orbit with energy E , either it behaves as described in the zero flux case, or

$$\lim_{t \rightarrow \infty} |\rho(t) - \rho_E(\theta(t))| = \lim_{t \rightarrow \infty} |\eta(t) - \eta_E(\theta(t))| = 0. \quad (1.4)$$

The interpretation of (1.4) is attraction to a spiral.

- iii) Above E_e , or for high enough energies if $\mathcal{E} = \emptyset$, *all* scattering orbits behave as described in the zero flux case.

Our knowledge about the low-energy region, $E < E_d$ if $\mathcal{E} \neq \emptyset$, is rather sparse. (This is also true for the case $b < 0$ of [CHS].)

The positive flux case may be reduced to the negative flux case by permutation symmetry.

1.2. Wave operators for a stable zero in quantum mechanics. Let us fix a stable zero, $b(\theta_j) = 0$, $b'(\theta_j) > 0$. We construct two types of wave operators 1) one for high energies, and 2) one for small energies. To define the relevant energy regimes we introduce the quantities

$$\beta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\frac{b'(\theta_j)}{\sqrt{2E}}}, \quad \tilde{\beta} = -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\frac{b'(\theta_j)}{\sqrt{2E}}}. \quad (1.5)$$

Those numbers arise as eigenvalues for a linearized reduced flow, cf. [HS2]. Now “high” means $4b'(\theta_j) < \sqrt{2E}$, while “small” means $\Re\beta < -3^{-1}$; in particular the two regimes fixed by these requirements, say $I_j = (E_j, \infty)$ and $\bar{I}_j = (0, \bar{E}_j)$ respectively, overlap.

1.2.1. High energy regime. We consider the equation for η from (1.3) (assuming $\rho > 0$)

$$\frac{d\eta}{d\theta} = -\frac{b + \eta}{\eta}\sqrt{2E - \eta^2},$$

now as a singular equation with θ nearby θ_j and the initial value $\eta(\theta_j) = 0$. Away from a discrete set of “resonances” there are smooth solutions η_E with the asymptotics $\eta_E \asymp \sqrt{2E}\beta(E)(\theta - \theta_j)$ as $\theta \rightarrow \theta_j$. Based on these functions one may construct local solutions $S = S(t, r, \theta) = S(t, x)$ to the Hamilton–Jacobi equation

$$h(x, \nabla_x S) = -\partial_t S.$$

By “local” we mean here that $|\theta - \theta_j|$ is small and that $2^{-1}(t^{-1}r)^2$ (or equivalently $E = -\partial_t S$) is well-localized (in particular bounded away from resonances) on the support of any such S .

We also need a quantity $w = w(t, r, \theta)$ which is defined as follows: First we notice that for any classical scattering orbit attracted to θ_j there exists the limit

$$w = \lim_{t \rightarrow \infty} r(t)^{-\beta}(\theta(t) - \theta_j).$$

We use this formula for the orbit that at time t goes through (r, θ) on the Lagrangian manifold associated with an S (in particular its energy is $E = -\partial_t S$).

Now the wave operator is given in terms of the family of local comparison dynamics

$$U_j(t) : L^2(I \times \mathbb{R}) \rightarrow L^2(\mathbb{R}_x^2) = L^2(\mathbb{R}_+ \times \mathbb{T}; r dr d\theta)$$

given as

$$[U_j(t)\phi](r, \theta) = e^{iS(t, r, \theta)} r^{-1/2} J_t^{1/2}(r, \theta) 1_{\{\cdot\}}(r, \theta) \phi(E(t, r, \theta), w(t, r, \theta)). \quad (1.6)$$

Here I ranges over small intervals free from resonances, $I \ni E(t, r, \theta) = -\partial_t S(t, r, \theta)$, J_t is the Jacobian determinant arising from the change of variables which makes $U_j(t)$ isometric and $1_{\{\cdot\}}$ projects to the region where $S(t, \cdot)$ is defined. In terms of a suitable (regularized) quantization H of the symbol h the wave operator is specified by its value at given $\phi \in L^2(I \times \mathbb{R})$ as

$$\Omega_j \phi = \lim_{t \rightarrow \infty} e^{itH} U_j(t) \phi.$$

We may view Ω_j as a uniquely defined (global) operator $\Omega_j : L^2(I_j \times \mathbb{R}) \rightarrow L^2(\mathbb{R}_x^2)$, see Remark 10.2 for details. Interpreted this way

$$H\Omega_j = \Omega_j M(E), \quad \text{Ran}(\Omega_j) \subseteq 1_{I_j}(H)L^2(\mathbb{R}_x^2);$$

here $M(E)$ denotes multiplication by E .

The above wave operator is closely related to the one at high energies for the potential model constructed recently in [HS1]. For earlier works on a somewhat similar “ x -space modifier” we refer to [Y] and [DG1].

1.2.2. *Small energy regime.* We may assume that $\theta_j = 0$. We use rectangular coordinates $x = (x_1, x_2)$, and similarly for the dual variable $\xi = (\xi_1, \xi_2)$. We substitute in the expression (1.1), Taylor expand up to second order in x_2/x_1 and ξ_2 , and replace x_1 by $t\xi_1$. The result reads

$$h(t) = 2^{-1}\xi^2 - \frac{b'(\theta_j)}{\xi_1} \frac{x_2}{t} \xi_2 + \frac{b'(\theta_j)}{\xi_1} \left(1 + \frac{b'(\theta_j)}{2\xi_1}\right) \left(\frac{x_2}{t}\right)^2.$$

Next we quantize this expression and obtain a family of self-adjoint operators $H(t)$, $t > 0$, which generates an explicit propagator

$$i\partial_t \bar{U}(t) = H(t)\bar{U}(t); \quad U(1) = I.$$

Let $\bar{J}_j = \left\{ \xi_1 : \xi_1 > 0, \frac{\xi_2^2}{2} \in \bar{I}_j \right\}$ and $p_1 = -i\frac{\partial}{\partial x_1}$. We prove the existence of the limit

$$\bar{\Omega}_j = s - \lim_{t \rightarrow \infty} e^{itH} \bar{U}(t) : 1_{\bar{J}_j}(p_1)L^2(\mathbb{R}_x^2) \rightarrow L^2(\mathbb{R}_x^2).$$

Again the wave operator diagonalizes the free energy

$$H\bar{\Omega}_j = \bar{\Omega}_j 2^{-1}p_1^2, \quad \text{Ran}(\bar{\Omega}_j) \subseteq 1_{\bar{I}_j}(H)L^2(\mathbb{R}_x^2).$$

A similar ‘‘Dollard-type’’ wave operator was considered at low energies for the potential model in [HS1].

1.3. **Wave operator for spirals.** Here we recall the construction of [CHS] adapted to the present case where the interval $\mathcal{E} = (E_d, E_e)$ of energies possessing solutions to (1.3) is bounded (assuming $\mathcal{E} \neq \emptyset$). Let

$$f(\theta) = \lim_{E \uparrow E_e} (\partial_E \rho_E(\theta))^{-1},$$

and

$$\mathcal{D} = \cup_{t>0} \{t\} \times \mathcal{D}_t; \quad \mathcal{D}_t = \{(r, \theta) \in \mathbb{R}_+ \times \mathbb{T} \mid 0 < \frac{r}{t} < f(\theta)\}. \quad (1.7)$$

We define on \mathcal{D}

$$S(t, r, \theta) = r\rho_{E(t, r, \theta)}(\theta) - tE(t, r, \theta), \quad (1.8)$$

where the ‘‘energy function’’ $E(t, \cdot, \theta)$ is the inverse of the function $\mathcal{E} \ni E \rightarrow r = t/(\partial_E \rho_E)(\theta) \in (0, tf(\theta))$.

Now we take as approximate dynamics the expression (1.6) modified as follows: Take $\phi \in L^2(\mathcal{E} \times \mathbb{T})$. Replace on the right hand side $\{\cdot\}$ by \mathcal{D}_t , and take $w(t, r, \theta)$ to be the uniquely defined initial angle, say θ_1 , for the orbit that at time t goes through $(r, \theta) \in \mathcal{D}_t$ on the Lagrangian manifold associated with S . More precisely given such pair (r, θ) there is a uniquely defined pair $(r_1, \theta_1) \in \mathcal{D}_1$ such that the direct flow (see (10.8)) starting at (r_1, θ_1) at time $t = 1$ ends up at (r, θ) at time t .

Let us denote this expression by $[U_{\text{sp}}(t)\phi](r, \theta)$. We can then define $\Omega_{\text{sp}}\phi = \lim_{t \rightarrow \infty} e^{itH} U_{\text{sp}}(t)\phi$. The range of the wave operator Ω_{sp} is a closed subspace of $L^2(\mathbb{R}^2)$ of states whose large time behaviour is ‘‘spiraling’’, see [CHS] for further discussion. Let P_{sp} denote the orthogonal projection onto this subspace. The wave operator diagonalizes the free energy

$$H\Omega_{\text{sp}} = \Omega_{\text{sp}}M(E), \quad \text{Ran}(\Omega_{\text{sp}}) \subseteq 1_{\mathcal{E}}(H)L^2(\mathbb{R}_x^2).$$

1.4. **Asymptotic completeness in quantum mechanics.** We would like to associate a quantum channel to each zero, $b(\theta_j) = 0$. Suppose $\mathcal{C} \subseteq (0, \infty)$ is an open set containing no eigenvalues of H . Then we would like to make sense of an expression like

$$P_j = P_{j, \mathcal{C}} = \lim_{f \uparrow 1_{\mathcal{C}}} \lim_{\chi_j \downarrow 1_{\{\theta_j\}}} \lim_{t \rightarrow \infty} e^{itH} \chi_j e^{-itH} f(H), \quad (1.9)$$

where f denotes a net of C_0^∞ -functions, and similarly χ_j is a net of operators given by multiplication by functions in $C^\infty(\mathbb{T})$. All limits are taken in the strong sense. If

it exists, P_j should project to a subspace $P_j L^2(\mathbb{R}_x^2) \subseteq 1_{\mathcal{C}}(H) L^2(\mathbb{R}_x^2)$. In the case of a stable zero, one would then ask for asymptotic completeness of the wave operators considered in Subsection 1.2, which for example for the entire high energy regime would mean that

$$\text{Ran}(\Omega_j) = P_{j, I_j \setminus \sigma_{pp}(H)} L^2(\mathbb{R}_x^2). \quad (1.10)$$

We shall justify the definition (1.9) and results like (1.10) in various cases.

1.4.1. *Completeness for the flux zero case.* Indeed if $\mathcal{C} = \mathcal{C}' \setminus \sigma_{pp}(H) \subseteq (0, \infty)$ is any open set of energies the definition (1.9) can be justified, and we show the decomposition into channels formula

$$1_{\mathcal{C}}(H) = \sum_j \oplus P_{j, \mathcal{C}}, \quad (1.11)$$

cf. [He]. In (1.11) only those projections which correspond to classically stable zeros are nonzero, cf. [HS2].

Moreover, high energy asymptotic completeness holds in the sense that for each stable zero

$$1_{\mathcal{C}'}(H) \text{Ran}(\Omega_j) = P_{j, \mathcal{C}'} L^2(\mathbb{R}_x^2); \quad \mathcal{C}' \subseteq I_j. \quad (1.12)$$

Thus in particular (1.10) follows.

Finally we show that the singular continuous spectrum is empty, and that the pure point spectrum is discrete in $(0, \infty) \setminus \mathcal{E}_{\text{exc}}$. Here \mathcal{E}_{exc} is either empty, finite or at most discrete in $(0, \infty)$; at those ‘‘exceptional’’ energies there exists a certain heteroclinic orbit for a reduced classical dynamics. We show

$$\sup \mathcal{E}_{\text{exc}} < 8^{-1} \left(\sup \int_{\theta_1}^{\theta_2} b d\theta \right)^2;$$

whence 0 is actually the only possible accumulation point. The (local) absence of ‘‘exceptional’’ orbits (see Condition 2.4 for definition, and Example 2.5 and Remark 2.6 for discussion) is used in our proof of the limiting absorption principle LAP in the zero flux case. Our proof relies on a partial differential equations scheme with a long history (and not Mourre theory, [Mo]). It is an open problem whether there is a Mourre estimate for small energies so that LAP would follow without need of Condition 2.4. If such an estimate exists, the conjugate operator would need to be somewhat sophisticated, cf. [ACH] and [CHS].

1.4.2. *Completeness for the general high energy region.* Without any condition on the flux nor existence of periodic solutions (i.e. $\mathcal{E} \neq \emptyset$) the results mentioned above for the flux zero case hold provided $\mathcal{C}' = (E', \infty) \cap I_j$ with E' taken large enough. This number may be given in terms of a condition on a reduced classical flow, see Condition 13.1. In the set (E', ∞) the singular continuous spectrum is empty and the pure point spectrum is discrete. For the case $\mathcal{E} \neq \emptyset$ we may take $E' = E_e$, cf. the discussion below.

1.4.3. *Completeness above E_d for $\mathcal{E} \neq \emptyset$.* We show that in (E_d, ∞) the singular continuous spectrum is empty and that the eigenvalues may only accumulate at E_d or ∞ . There are two regions 1) above E_e and 2) in \mathcal{E} . As for 1) the previous results hold with $\mathcal{C}' = (E_e, \infty) \cap I_j$ (and similarly for $\mathcal{C}' = (E_e, \infty) \cap \bar{I}_j$ if $E_e < \bar{E}_j$). As for 2) we take $\mathcal{C}' = \mathcal{E}$. Again we can justify (1.9). As for (1.11) we have the substitute

$$1_{\mathcal{C}}(H) = P_{\text{sp}} \oplus \sum_j \oplus P_{j, \mathcal{C}}, \quad (1.13)$$

where as above only stable zeros contribute to the second summation. In an obvious way the ranges of the projections $P_{j, \mathcal{C}}$ of (1.13) may be identified in terms of the

ranges of the wave operators, cf. (1.12) (hence completeness holds). Here we need to use both of the wave operators Ω_j and $\bar{\Omega}_j$ unless either $\mathcal{E} \subseteq I_j$ or $\mathcal{E} \subseteq \bar{I}_j$.

1.5. Organization of paper. The paper is organized as follows: We include throughout the paper a possibly non-trivial scalar potential depending only on the angle. We could also have included faster decaying fields possibly with local singularities, but for simplicity of presentation we have put the discussion on such rather trivial inclusions in a remark, see Remark 6.6.

In Section 2 we introduce notation, basic equations and assumptions, and two conditions. We state a basic result for the classical system under the zero flux condition, Condition 2.1. Further details and study of the classical system with or without zero flux are deferred to Section 12.

The wave operators of Subsection 1.2 are explained in more detail in Section 10. As for the one in Subsection 1.3 we refer to [CHS] for a thorough account.

In Section 3 we introduce the Hamiltonian, and we study various preliminaries for LAP in particular a version of Hörmander's propagation of singularities theorem [Hö2, Proposition 3.5.1]. (To our knowledge Melrose was the first who applied this theorem in scattering theory, see [Melr] and [HMV].)

All of the sections, Sections 4–9 and Section 11, are devoted to quantum mechanics under Condition 2.1. In Sections 4–6 we prove LAP. In Section 7 we use LAP to show various other basic estimates all of which have a clear classical interpretation. In Section 8 we introduce a channel projection P_j for any given fixed point (stable or unstable), cf. (1.9). (It is slightly more complicated due to a possible energy-dependence of the fixed point angle in the general case.) We show some basic estimates for states associated to such a channel. In Section 9 we show that the projections for the unstable fixed points do not contribute to the decomposition of channels formula, cf. (1.11). Finally, Section 11 is devoted to showing asymptotic completeness for the wave operators of Section 10.

In Section 13 we study LAP and completeness in the case of negative flux. We give an outline of how the methods used in the zero flux case may be modified under some conditions stated in this section. To keep this paper “short” we leave out most details, in particular those that are closely related to [CHS].

In Appendix A we introduce collapsing classical orbits which correspond to orbits in the configuration space with a singularity at $x = 0$. We investigate the smallness and general nature of the set of collapsing orbits. Our results are not needed in our treatment of the quantum model, however they are a natural supplement to Section 12 with some interest of their own. The last part of Appendix A is devoted to proving discreteness of the set \mathcal{E}_{exc} of exceptional energies.

In Section B we shall briefly discuss another model exhibiting similar properties as the ones for the model of the paper. This model is from Riemannian geometry.

2. ASSUMPTIONS, CLASSICAL MECHANICS AND THE ZERO FLUX CASE

We shall study symbols of the form

$$h = 2^{-1}(\xi - \mathbf{a} - \mathbf{a}_\delta)^2 + V + V_\delta, \quad (2.1)$$

where $\mathbf{a} = \mathbf{a}(x) = \mathbf{a}(\theta) = (-\sin \theta, \cos \theta)b(\theta)$, $V = V(x) = V(\theta)$, and $\mathbf{a}_\delta = \mathbf{a}_\delta(x)$ and $V_\delta(x)$ have decay (measured by some $\delta > 0$). For convenience we assume below that $\mathbf{a}_\delta = 0$ and $V_\delta = 0$, and devote Remark 6.6 to a discussion on inclusion of non-trivial perturbations. We assume that b and V are real-valued, smooth and 2π -periodic functions of the angle θ .

We may write

$$h = 2^{-1}(\rho^2 + \eta^2) + V,$$

where

$$\rho = \hat{x} \cdot \xi, \quad \hat{x} = (\cos \theta, \sin \theta) = x/r, \quad r = |x|, \quad (2.2)$$

$$\eta = (-\sin \theta, \cos \theta) \cdot \xi - b. \quad (2.3)$$

Let us fix an open interval $I \subseteq (\min V, \infty)$ free from critical values, i.e.

$$V(\theta) = E \in I \Rightarrow V'(\theta) \neq 0. \quad (2.4)$$

The equations of motion in the “new time”

$$\tau = \int r^{-1} dt, \quad (2.5)$$

read

$$\begin{cases} \frac{d}{d\tau} \theta = \eta \\ \frac{d}{d\tau} \eta = -(\eta + b)\rho - V' \\ \frac{d}{d\tau} \rho = (\eta + b)\eta \end{cases}. \quad (2.6)$$

We notice that this system is complete in the sense that the maximal solutions are defined for all $\tau \in \mathbb{R}$. Clearly the energy h is a preserved observable.

Obviously the fixed points are given by the two systems of equations

$$\eta = b\rho + V' = 0, \quad (2.7)$$

$$\rho = +\sqrt{2(h - V)} \text{ or } \rho = -\sqrt{2(h - V)}. \quad (2.8)$$

The set of fixed points $z^+ = z = (\theta, \eta, \rho)$ for which $\rho > 0$ and $h \in I$ will be denoted by \mathcal{F}^+ . The set of fixed points $z^- = (\theta, \eta, \rho)$ for which $\rho < 0$ and $h \in I$ will be denoted by \mathcal{F}^- . Let $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$. For $E \in I$ we put $\mathcal{F}^+(E) = \mathcal{F}^+ \cap \{h = E\}$, and introduce similarly sets $\mathcal{F}^-(E)$ and $\mathcal{F}(E)$. With the conditions

$$\kappa^+ := \frac{d}{d\theta}(b\sqrt{2(E - V)} + V') \neq 0 \text{ at all } z^+ \in \mathcal{F}^+(E), \quad (2.9)$$

and

$$\kappa^- := \frac{d}{d\theta}(-b\sqrt{2(E - V)} + V') \neq 0 \text{ at all } z^- \in \mathcal{F}^-(E), \quad (2.10)$$

there is at most a finite number of points in $\mathcal{F}(E)$. Moreover they depend smoothly on E .

For simplicity of presentation we shall tacitly assume (2.4), (2.9) and (2.10) throughout the paper unless otherwise stated (there are exceptions in Section 12).

For any given $z^+ = (\theta_j, 0, \rho_j) \in \mathcal{F}^+(E)$ we may eliminate ρ in the second equation of (2.6) near z^+ (by the energy relation) and obtain an autonomous system whose linearization at the fixed point $(\theta, \eta) = (\theta_j, 0)$ takes the following form: For $y = (\theta - \theta_j, \eta)^{\text{tr}}$

$$\frac{d}{d\tau} y = A^+ y; \quad A^+ = \begin{pmatrix} 0 & 1 \\ -\kappa^+ & -\sqrt{2(E - V)} \end{pmatrix}.$$

Here A^+ is evaluated at $(E, \theta_j(E))$, that is $A^+ = A^+(E, \theta_j(E))$.

The eigenvalues of A^+ are given by

$$\lambda^+ = -2^{-1}\sqrt{2(E - V)} + (-)2^{-1}\sqrt{2(E - V) - 4\kappa^+}. \quad (2.11)$$

We proceed similarly at $z^- \in \mathcal{F}^-(E)$; the eigenvalues of the corresponding matrix A^- are given by

$$\lambda^- = 2^{-1}\sqrt{2(E - V)} + (-)2^{-1}\sqrt{2(E - V) - 4\kappa^-}. \quad (2.12)$$

We assume for convenience that for $z \in \mathcal{F}(E)$

$$2(E - V) \neq 4\kappa^\sharp. \quad (2.13)$$

With (2.13) the matrix A^\sharp is diagonalized by the matrix

$$T(E, z^\sharp(E)) = \begin{pmatrix} 1 & 1 \\ \lambda_1^\sharp & \lambda_2^\sharp \end{pmatrix},$$

where \sharp refers to either plus or minus and the subscript refers to an arbitrary numbering of the two eigenvalues in question at $z^\sharp = z^\sharp(E)$. The observable

$$l = l_E = |T(E, z^\sharp(E))^{-1} \bar{z}|_{\mathbb{C}}^2; \quad \bar{z} = (\theta - \theta^\sharp(E), \eta)^{\text{tr}}, \quad (2.14)$$

is a ‘‘Liapunov-function’’ near $z^\sharp(E)$ if the fixed point is a sink (where l is decreasing) or a source (where l is increasing); that is, if the real part of the two eigenvalues has the same sign. If (2.13) is violated one may still construct a similar smooth function $l = l_E(\theta, \eta)$, see [HS2], and this would be a substitute for (2.14) in our applications.

We denote by $\mathcal{F}_{\text{sa}}^+$ the set of saddle points in \mathcal{F}^+ (corresponding to having $\kappa^+ < 0$ in (2.11)). Similarly $\mathcal{F}_{\text{si}}^+ = \mathcal{F}^+ \setminus \mathcal{F}_{\text{sa}}^+$, $\mathcal{F}_{\text{sa}}^-$ is the set of saddle points in \mathcal{F}^- and $\mathcal{F}_{\text{so}}^- = \mathcal{F}^- \setminus \mathcal{F}_{\text{sa}}^-$. Furthermore we put $\mathcal{F}_{\text{sa}}^+(E) = \mathcal{F}_{\text{sa}}^+ \cap \{h = E\}$, etc.

Condition 2.1 (Zero flux condition).

$$\int_0^{2\pi} b(\theta) d\theta = 0. \quad (2.15)$$

We introduce

$$\tilde{b}(\theta) := \int_0^\theta b(\varphi) d\varphi, \quad \theta \in \mathbb{R}. \quad (2.16)$$

With (2.15) the function \tilde{b} is 2π -periodic.

Lemma 2.2. *Suppose $E \in I$ obeys $E < \max V$, then $\mathcal{F}^+(E)$ and $\mathcal{F}^-(E)$ are non-empty.*

Suppose Condition 2.1 and $E \in I$ obeys $E > \max V$, then all of the sets $\mathcal{F}_{\text{sa}}^+(E)$, $\mathcal{F}_{\text{si}}^+(E)$, $\mathcal{F}_{\text{sa}}^-(E)$ and $\mathcal{F}_{\text{so}}^-(E)$ are non-empty.

Proof. As for the first part of the lemma we pick θ_1 such that $V(\theta_1) = \min V$ and a maximal interval $J = (\theta_1^-, \theta_1^+) \ni \theta_1$ such that $V(\theta) < E$ for $\theta \in J$. Look at $f : J \rightarrow \mathbb{R}$ given by $f = b\sqrt{2(E - V)} + V'$. Since $\lim_{\theta \rightarrow \theta_1^-} f(\theta) < 0$ and $\lim_{\theta \rightarrow \theta_1^+} f(\theta) > 0$ there exists $\theta \in J$ such that $f(\theta) = 0$. This shows that $\mathcal{F}^+(E) \neq \emptyset$. We may argue similarly for $\mathcal{F}^-(E)$.

As for the second part of the lemma we prove that $\mathcal{F}^+(E)$ has at least two elements for $E > \max V$. We need to show that $b(\theta)\sqrt{2(E - V(\theta))} + V'(\theta)$ has at least two zeroes on the torus. But this is the same as showing that

$$b + \frac{V'}{\sqrt{2(E - V(\theta))}} = \left(\tilde{b}(\cdot) - \sqrt{2(E - V(\theta))} \right)'$$

has at least two zeroes, which is implied by the fact that this function is periodic and its integral on the torus is zero. Clearly by (2.9) one of the zeros obeys $\kappa^+ > 0$ while another one obeys $\kappa^+ < 0$. We may argue similarly for $\mathcal{F}^-(E)$. \square

We notice that the second part of Lemma 2.2 is in agreement with the Poincaré-Hopf theorem since for $E > \max V$ the corresponding energy surface is topological a torus. Below $\max V$ the situation is different, in fact there might be no saddle points at all.

Proposition 2.3. *Suppose Condition 2.1. Then every integral curve $\gamma(\tau) = (\theta(\tau), \eta(\tau), \rho(\tau))$ of the system (2.6) with energy $h = E \in I$ has the property that there exist $z_1, z_2 \in \mathcal{F}(E)$ such that*

$$\lim_{\tau \rightarrow +\infty} \gamma(\tau) = z_1. \quad (2.17)$$

and

$$\lim_{\tau \rightarrow -\infty} \gamma(\tau) = z_2. \quad (2.18)$$

Proof. Let $y = (b\rho + V', \eta)^{\text{tr}}$. Using (2.6) we compute for any classical orbit $y = y(\tau)$

$$\frac{d}{d\tau} y = Ay + \mathcal{O}(\eta^2), \quad (2.19)$$

where

$$A = \begin{pmatrix} 0 & b'\rho + b^2 + V'' \\ -1 & -\rho \end{pmatrix}.$$

Also we introduce

$$\begin{aligned} a_1(\tau) &= \rho(\tau) - \tilde{b}(\theta(\tau)) \\ a_2(\tau) &= Ca_1(\tau) - 2y_1(\tau)y_2(\tau). \end{aligned} \quad (2.20)$$

We differentiate a_1 and a_2 using (2.6) and (2.19) and get

$$a_1' = \eta^2, \quad (a_2 - Ca_1)' \geq q + \mathcal{O}(\eta^2); \quad q = y^2. \quad (2.21)$$

Taking $C > 0$ large enough we deduce that

$$\frac{d}{d\tau} a_2 \geq q. \quad (2.22)$$

Notice that a_1 is always bounded due to the zero flux condition. This implies that a_2 is also bounded. Whence by integrating a_2' we infer that

$$\int_0^\infty q(\tau) d\tau < \infty. \quad (2.23)$$

By the finiteness of the integral in (2.23) there exists a sequence $\tau_n \rightarrow +\infty$ such that $q(\tau_n) \rightarrow 0$. Hence if we prove that $\lim_{\tau \rightarrow +\infty} q(\tau)$ exists, it must be zero. In order to achieve this, we employ a Cook-type argument, that is, we prove that $q'(\cdot) \in L^1([0, \infty))$. The verification is obvious since from (2.19) we get

$$|q'(\tau)| \leq \text{const} \cdot q(\tau), \quad \tau \geq 0,$$

so using again (2.23) we get the result.

We can therefore conclude that

$$\lim_{\tau \rightarrow +\infty} [b(\theta(\tau))\rho(\tau) + V'(\theta(\tau))] = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} \eta(\tau) = 0. \quad (2.24)$$

From this and compactness we conclude that along some sequence $\tau_n \rightarrow +\infty$,

$$\theta \rightarrow \theta_1, \quad \eta \rightarrow 0, \quad \rho \rightarrow \rho_1. \quad (2.25)$$

Here either, $\rho_1 = \sqrt{2(E - V(\theta_1))}$ or $\rho_1 = -\sqrt{2(E - V(\theta_1))}$. In particular $z_1 = (\theta_1, 0, \rho_1) \in \mathcal{F}(E)$. Using (2.9) and (2.10) we see that the convergence $z \rightarrow z_1$ of (2.25) in fact holds for $\tau \rightarrow +\infty$ rather than along the sequence $\tau_n \rightarrow +\infty$.

We may argue similarly for $\tau \rightarrow -\infty$. \square

For a generalization of Proposition 2.3 we refer to Proposition 12.2.

To deal with quantum mechanics we shall need the following condition.

Condition 2.4 (No exceptional orbits condition). An energy $E \in \mathbb{R}$ is said to be exceptional if there exists an (exceptional) integral curve $\gamma(\cdot)$ of the system (2.6) with $h = E$ such that simultaneously

$$\lim_{\tau \rightarrow +\infty} \gamma(\tau) \in \mathcal{F}_{\text{sa}}^- \quad \text{and} \quad \lim_{\tau \rightarrow -\infty} \gamma(\tau) \in \mathcal{F}_{\text{sa}}^+. \quad (2.26)$$

Denoting by \mathcal{E}_{exc} the set of exceptional energies, $\mathcal{E}_{\text{exc}} \cap I = \emptyset$.

We remark that Condition 2.4 is fulfilled if $b = 0$, since in this case the observable $e^{\int \rho}$ is increasing in τ .

Example 2.5 (Criterion at high energies). Suppose $V = 0$. For all integral curves γ with energy $E > 0$ we may pick $\epsilon = \epsilon(E) > 0$ such that along γ

$$\frac{d}{d\tau} a_2 \geq \frac{\epsilon}{2}(\eta^2 + b^2), \quad (2.27)$$

cf. (2.20) and (2.22). Let $C(E) = \epsilon^{-1}C$. We claim that (2.26) is not possible for

$$E > 8^{-1} \left((1 - C(E)^{-1}) \sup[\tilde{b}] \right)^2; \quad [\tilde{b}]_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} b(\theta) d\theta. \quad (2.28)$$

To see this let us assume that there is an exceptional orbit starting at $\theta = \theta_1$ and ending at $\theta = \theta_2$. Then we may use

$$-[\tilde{b}]_{\theta_1}^{\theta_2} = - \int_{-\infty}^{\infty} b\eta d\tau$$

and (2.27) to estimate

$$-[\tilde{b}]_{\theta_1}^{\theta_2} \leq \epsilon^{-1} [a_2]_{\theta_1}^{\theta_2} = C(E) (-2\sqrt{2E} - [\tilde{b}]_{\theta_1}^{\theta_2}),$$

yielding

$$2\sqrt{2E} \leq -(1 - C(E)^{-1}) [\tilde{b}]_{\theta_1}^{\theta_2}.$$

We remark that the observable $a = r(\rho - \tilde{b} + c)$, $c = 2^{-1}(\max \tilde{b} + \min \tilde{b})$, provides a Mourre estimate for energies $E > 8^{-1}(\sup[\tilde{b}])^2$ in this case. Thus the bound (2.28) is a slight improvement.

Remark 2.6. One obtains “ r ” from the system (2.6) by the formula $r = r_0 \exp(\int \rho d\tau)$. In particular any exceptional orbit defines a one-parameter family of classical orbits in the full phase space. This may also be seen from the invariance of Newton’s equation, $x(t) \rightarrow cx(c^{-1}t)$ for $c > 0$. The configuration part $x(t)$ of any solution associated with an exceptional orbit has the peculiar property that it collapses at the origin both backwards and forwards in finite “physical” time t . (In other words any maximal solution is only defined on a bounded interval.) In Appendix A we show that $\mathcal{E}_{\text{exc}} \cap (\max V, \infty)$ is discrete in $(\max V, \infty)$.

For a more general discussion of collapsing orbits we also refer the reader to Appendix A.

Throughout the paper the notation $F(x > \epsilon)$ denotes a smooth increasing function $= 1$ for $x > \frac{3}{4}\epsilon$ and $= 0$ for $x < \frac{1}{2}\epsilon$; $F(\cdot < \epsilon) := 1 - F(\cdot > \epsilon)$.

3. PROPAGATION OF SINGULARITIES

For the results of this section, Conditions 2.1 and 2.4 are not used.

Let $L^{2,s} = X^{-s}L^2(\mathbb{R}_x^2)$, $X = \langle x \rangle = (1 + |x|^2)^{1/2}$. Introduce $L^{2,-\infty} = \cup_{s \in \mathbb{R}} L^{2,s}$ and $L^{2,\infty} = \cap_{s \in \mathbb{R}} L^{2,s}$. We define the wave front set $WF^s(u)$ of a distribution $u \in L^{2,-\infty}$ to be the subset of $\mathbb{T} \times \mathbb{R}_\xi^2$ whose complement is given by

$$\begin{aligned} (\theta_0, \xi_0) = z_0 &\notin WF^s(u) \\ \Leftrightarrow & \\ \exists \text{ neighborhoods } \mathcal{N}_{\theta_0} \ni \theta_0, \mathcal{N}_{\xi_0} \ni \xi_0 \forall \chi_{\theta_0} \in C^\infty(\mathcal{N}_{\theta_0}), \chi_{\xi_0} \in C_0^\infty(\mathcal{N}_{\xi_0}) : & \\ \chi_{\theta_0}(\theta) \chi_{\xi_0}(p) u \in L^{2,s}. & \end{aligned} \quad (3.1)$$

Here $p = \mathbf{p} = -i\nabla$.

For $z \in \mathbb{T} \times \mathbb{R}^2$

$$L_z^{2,s} = \{u \in L^{2,-\infty} | z \notin WF^s(u)\}.$$

Obviously

$$WF^s(u) = \emptyset \Leftrightarrow \forall z \in \mathbb{T} \times \mathbb{R}^2 : u \in L_z^{2,s},$$

and

$$u \in L^{2,s} \Rightarrow WF^s(u) = \emptyset.$$

Conversely (by a compactness argument), if for some $\chi \in C_0^\infty(\mathbb{R}^2)$

$$u - \chi(p)u \in L^{2,s},$$

then

$$WF^s(u) = \emptyset \Rightarrow u \in L^{2,s}.$$

We recall the continuous inclusions for Besov spaces

$$L^{2,-1/2} \subset B_0^* \subset B^* \subset L^{2,-1/2-\epsilon}; \quad \epsilon > 0, \quad (3.2)$$

cf. [Hö1, Section 14.1]. The norm on B^* may be taken as

$$\|u\|_{B^*}^2 = \sup_{R>1} R^{-1} \int_{|x|<R} |u|^2 dx,$$

and $u \in B_0^*$ if and only if also

$$\lim_{R \rightarrow \infty} R^{-1} \int_{|x|<R} |u|^2 dx = 0.$$

We shall use the standard metric

$$g = \langle x \rangle^{-2} dx^2 + d\xi^2,$$

and standard Weyl quantization $\text{Op}^w(a)$ of symbols $a \in S(m, g)$. Here we shall use m of the form $m = \langle x \rangle^s \langle \xi \rangle^t$.

Now let us look at the quantization H of (2.1). We shall essentially assume that $\mathbf{a}_\delta = 0$ and $V_\delta = 0$ (see Remark 6.6 for an extension), but for convenience we cut off the singularity at the origin. Whence we consider the symbol \bar{h} obtained by replacing \mathbf{a} and V in the expression h by $F(r > 1)\mathbf{a}$ and $F(r > 1)V$, respectively. A small computation shows that

$$H := \text{Op}^w(\bar{h}) = 2^{-1}(\mathbf{p} - F(r > 1)\mathbf{a})^2 + F(r > 1)V. \quad (3.3)$$

We shall prove a propagation of singularity result using (2.2) and (2.3) to identify $\mathbb{T} \times \mathbb{R}_\xi^2$ as

$$\mathbb{T}^* := \mathbb{T} \times \mathbb{R}_{(r,\rho)}^2.$$

Thus \mathbb{T}^* is the phase space of the system (2.6) (and not the cotangent bundle of \mathbb{T} nor the phase space of the full Hamiltonian system $T^*\mathbb{R}^2$).

If $(H - E)u = v$ for $E \in I$ and $v \in L^{2,s}$, then (by ellipticity)

$$WF^s(u) \subseteq \mathbb{T}_E^* := \{z \in \mathbb{T}^* \mid h(z) = E\}. \quad (3.4)$$

The maximal solution of the system (2.6) that passes z at $\tau = 0$ is denoted by $\gamma(\tau, z)$ or $\phi_\tau(z)$.

Proposition 3.1. *Suppose $u \in L^{2,s-1}$, $v \in L^{2,s+1}$, $E \in I$ and $(H - E)u = v$. Then*

$$\gamma(\mathbb{R} \times (\mathbb{T}^* \setminus WF^s(u))) \subseteq \mathbb{T}^* \setminus WF^s(u). \quad (3.5)$$

Proof. Our proof is a modification of that of [Hö2, Proposition 3.5.1], see also [Melr] and [HMV]. Suppose $u \in L_z^{2,s}$ for some $z \in \mathbb{T}^*$. Then we need to show that $u \in L_{\gamma(\tau)}^{2,s}$ along the integral curve $\gamma(\cdot) = \gamma(\cdot, z)$. Consider

$$\tau_0 = \sup\{\tau > 0 \mid u \in L_{\gamma(\tilde{\tau})}^{2,s} \text{ for all } \tilde{\tau} \in [0, \tau]\}. \quad (3.6)$$

We need to show that $\tau_0 = \infty$. (Similar arguments will work in the backward direction.)

Suppose on the contrary that τ_0 is finite. Then $\gamma(\tau_0)$ is not a fixed point. Consequently we may pick a slightly smaller $\tau'_0 < \tau_0$ and a transversal 2-dimensional

submanifold at $\gamma(\tau'_0)$, say \mathcal{M} , such that with $I = (-\epsilon + \tau'_0, \tau_0 + \epsilon)$ for some $\epsilon > 0$ the map

$$I \times \mathcal{M} \ni (\tau, m) \rightarrow \Psi(\tau, m) = \gamma(\tau - \tau'_0, m) \in \mathbb{T}^*,$$

is a diffeomorphism onto its range.

We pick $\chi \in C_0^\infty(\mathcal{M})$ supported near $\gamma(\tau'_0)$ such that

$$\chi(\gamma(\tau'_0)) = 1.$$

Since $u \in L_{\gamma(\tau'_0)}^{2,s}$ we can find a non-positive function $f \in C_0^\infty(I)$ such that $f' \geq 0$ on a neighborhood of $[\tau'_0, \tau_0 + \epsilon)$, $f(\tau_0) < 0$ and the set

$$\Psi((-\epsilon + \tau'_0, \tau'_0] \times \text{supp } \chi) \cap WF^s(u) = \emptyset. \quad (3.7)$$

Let $f_K(\tau) = \exp(-K\tau)f(\tau)$ for $K > 0$, and $X_\kappa = (1 + \kappa r^2)^{1/2}$ for $\kappa \in (0, 1)$. We consider the classical observable

$$b_\kappa = X^{1/2}a_\kappa; \quad a_\kappa = X^s X_\kappa^{-3/2} F(r > 1)(f_K \otimes \chi) \circ \Psi^{-1}. \quad (3.8)$$

First we fix K : A part of the Poisson bracket with b_κ^2 is

$$\{h, X^{2s+1}X_\kappa^{-3}\} = X^{-1}(Y_\kappa \rho)X^{2s+1}X_\kappa^{-3}, \quad (3.9)$$

where $Y_\kappa = Y_\kappa(r)$ is uniformly bounded in κ . We fix K such that $2K \geq |Y_\kappa \rho| + 2$ on $\text{supp } b_\kappa$.

By the system (2.6)

$$\{h, (f_K \otimes \chi) \circ \Psi^{-1}\} = r^{-1}([\frac{d}{d\tau} f_K] \otimes \chi) \circ \Psi^{-1}. \quad (3.10)$$

From (3.9) and (3.10), and by the choice of f and K , we conclude that

$$\{h, b_\kappa^2\} \leq -2a_\kappa^2 \text{ at } \mathcal{P} \quad (3.11)$$

given by

$$\mathcal{P} = [1, \infty) \times \Psi(\{\tau | f'(\tau) \geq 0\} \times \text{supp } \chi),$$

and therefore in particular

$$\{h, b_\kappa^2\}(r, \gamma(\tau_0)) < 0. \quad (3.12)$$

Next we introduce $A_\kappa = \text{Op}^w(a_\kappa)$ and $B_\kappa = \text{Op}^w(b_\kappa)$. We write

$$\langle i[H, B_\kappa^2] \rangle_u = -2\mathfrak{S}\langle v, B_\kappa^2 u \rangle, \quad (3.13)$$

and estimate the right hand side using the calculus of pseudodifferential operators [Hö1, Theorems 18.5.4, 18.6.3] and the Cauchy-Schwarz inequality to obtain the uniform bound

$$|\langle i[H, B_\kappa^2] \rangle_u| \leq C_1 \|v\|_{s+1} \|A_\kappa u\| + C_2 \leq \|A_\kappa u\|^2 + C_3. \quad (3.14)$$

On the other hand using (3.3), (3.7), (3.11) and the calculus [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8] we infer that

$$\langle i[H, B_\kappa^2] \rangle_u \leq -2\|A_\kappa u\|^2 + C_4. \quad (3.15)$$

Combining (3.14) and (3.15) yields

$$\|A_\kappa u\|^2 \leq C_5 = C_3 + C_4,$$

which in combination with (3.12) in turn yields a uniform bound

$$\|X_\kappa^{-1} \chi_{\gamma(\tau_0)} u\|_s^2 \leq C_6. \quad (3.16)$$

Here $\chi_{\gamma(\tau_0)}$ signifies a non-trivial phase-space localization factor of the form entering in (3.1) centered at the point $\gamma(\tau_0)$ (using polar coordinates).

We let $\kappa \rightarrow 0$ in (3.16) and infer that $u \in L_{\gamma(\tau_0)}^{2,s}$, which is a contradiction. \square

Remark 3.2. The assumption $u \in L^{2,s-1}$ of Proposition 3.1 is not needed. This may be shown by iterating the method of proof, cf. [Hö2, Proposition 3.5.1].

An elaboration of the above proof gives the following result, cf. [Hö2, Proposition 3.5.1]. For $z \in \mathbb{T}^*$ we use the notation \mathcal{N}_z to denote a neighborhood of z , and for $\chi \in C_0^\infty(\mathcal{N}_z)$ the notation $\text{Op}^w(\chi)$ for the operator $\text{Op}^w(a)$ with symbol $a(x, \xi) = F(r > 1)\chi$ as defined by the identification (2.2) and (2.3).

Proposition 3.3. *Suppose $u \in L^{2,s-1}$, $v \in L^{2,s+1}$, $\Re\zeta \in I$, $\Im\zeta \geq 0$ and $(H - \zeta)u = v$. Suppose $z_1, z_2 \in \mathbb{T}^*$ are linked as $\gamma(\tau_0, z_1) = z_2$ for a positive τ_0 . Then*

$$\begin{aligned} \forall \mathcal{N}_{z_1} \exists \mathcal{N}_{z_2} \forall \chi_2 \in C_0^\infty(\mathcal{N}_{z_2}) \exists \chi_1 \in C_0^\infty(\mathcal{N}_{z_1}) : \\ \|\text{Op}^w(\chi_2)u\|_s^2 \leq C (\|u\|_{s-1}^2 + \|\text{Op}^w(\chi_1)u\|_s^2 + \|v\|_{s+1}^2). \end{aligned} \quad (3.17)$$

The bound (3.17) is uniform in ζ in a bounded set.

Remark 3.4. Under the condition of Proposition 3.1 the sign condition in Proposition 3.3 is redundant. Thus for ζ real the wave front bound (3.17) is valid in both directions of the flow.

Proposition 3.5. *Suppose $u \in L^{2,-1}$, $E \in I$ and $(H - E)u = 0$. Suppose $WF^{-\frac{1}{2}}(u) \subseteq \mathbb{T}_+^* = \{z \in \mathbb{T}^* | \rho > 0\}$. Then*

$$u \in B_0^*. \quad (3.18)$$

Proof. We follow the proof of [Hö1, Theorem 30.2.6]. Pick a real-valued decreasing $\psi \in C_0^\infty([0, \infty))$ such that $\psi(r) = 1$ in a small neighborhood of 0 and $\psi'(r) = -1$ for $1/2 \leq r \leq 1$. Let $\psi_R(x) = \psi(r/R)$ for $R > 1$.

$$0 = \langle i[H - E, \psi_R] \rangle_u = 2^{-1} R^{-1} \Re \langle (\hat{x} \cdot p + p \cdot \hat{x}) \psi'(r/R) \rangle_u. \quad (3.19)$$

We use (3.2) and the assumption of the proposition to bound the right hand side of (3.19) as

$$\dots \leq \epsilon R^{-1} \langle \psi'(r/R) \rangle_u + T_R,$$

where

$$\lim_{R \rightarrow \infty} T_R = 0.$$

We conclude (3.18). \square

4. MICROLOCAL BOUNDS

In this section Condition 2.1 (but not Condition 2.4) is imposed. Suppose we have given $u \in L^2$ and $v \in L^{2,t}$ such that $(H - \zeta)u = v$, where $t \in (1/2, 1)$, $\Re\zeta \in I$ and $\Im\zeta \geq 0$, cf. Proposition 3.3.

4.1. Bounds at $\mathcal{F}_{\text{so}}^-$. We shall prove microlocal bounds of u at $\mathcal{F}_{\text{so}}^-$ (the set of “sources”) somewhat along the line of the proof of Proposition 3.1. Consider a branch $z^-(\cdot) \in \mathcal{F}_{\text{so}}^-$. Instead of the observable (3.8) we now consider

$$b_\kappa = X^{1/2} a_\kappa; \quad a_\kappa = \left(\frac{X}{X_\kappa}\right)^{t-1} X_\kappa^{-1/2} F(r > 1) f(h) F(l < \epsilon), \quad (4.1)$$

where $f \in C_0^\infty(I)$, f real-valued, and $l = l_h(\theta, \eta)$ is the “Liapunov-function” as defined in (2.14), in the present context in terms of z^- .

Picking $\epsilon > 0$ small enough in (4.1) we may estimate part of the Poisson bracket with b_κ^2 as

$$\dots \{h, F(l < \epsilon)\} \leq \delta r^{-1} F'(l < \epsilon), \quad (4.2)$$

for some positive δ (which essentially is determined as a uniform lower bound of the real part of the eigenvalues (2.12)).

Obviously on the support of b_κ^2

$$\{h, \left(\frac{X}{X_\kappa}\right)^{2t-1}\} = (1 - \kappa)(2t - 1)r X^{-1} X_\kappa^{-3} \rho \left(\frac{X}{X_\kappa}\right)^{2t-2} < 0. \quad (4.3)$$

We quantize (4.1) writing $A_\kappa = \text{Op}^w(a_\kappa)$ and $B_\kappa = \text{Op}^w(b_\kappa)$. Using (4.2) and (4.3) we may estimate

$$\langle i[H, B_\kappa^2] \rangle_u \leq -2\sigma \langle X_\kappa^{-2} \rangle_{A_\kappa u} + C_1 \|u\|_{-1}^2. \quad (4.4)$$

On the other hand since $(\Im \zeta) B_\kappa^2 \geq 0$

$$\begin{aligned} \langle i[H - \zeta, B_\kappa^2] \rangle_u &\geq 2\Im \langle B_\kappa^2 u, v \rangle \\ &\geq -\sigma \|X_\kappa^{1/2-t} A_\kappa u\|^2 - C_2 \|u\|_{-1}^2 - C_3 \|v\|_t^2. \end{aligned} \quad (4.5)$$

We combine (4.4) and (4.5) to obtain

$$2\sigma \|X_\kappa^{-1} A_\kappa u\|^2 - \sigma \|X_\kappa^{1/2-t} A_\kappa u\|^2 \leq (C_1 + C_2) \|u\|_{-1}^2 + C_3 \|v\|_t^2, \quad (4.6)$$

from which we conclude by letting $\kappa \rightarrow 0$ (and dividing by σ) that

$$\begin{aligned} \|\tilde{A}u\|_{t-1}^2 &\leq C_4 \|u\|_{-1}^2 + C_5 \|v\|_t^2; \\ \tilde{A} &= \text{Op}^w(F(r > 1)f(h)F(l < \epsilon)), \end{aligned} \quad (4.7)$$

with constants independent of ζ in a bounded set.

We notice that a similar bound at $\mathcal{F}_{\text{sa}}^-$ would require an extra input. We will come back to that point in Section 5.

4.2. Bounds away from \mathcal{F} . Consider the quantization $A_2 = \text{Op}^w(\bar{a}_2)$ of the symbol $\bar{a}_2 = F(r > 1)f(h)a_2$ where a_2 is given by (2.20) and the other factors are as in (4.1). Introduce

$$e = r^{-1}F(r > 1)f(h)((b\rho + V')^2 + \eta^2) (\in S(\langle x \rangle^{-1}, g)). \quad (4.8)$$

We compute

$$i[H, A_2] \geq \text{Op}^w(e) + \text{Op}^w(r_{-3}), \quad (4.9)$$

where $r_{-3} \in S(\langle x \rangle^{-3}, g)$. Since A_2 is bounded we may pick a constant C_2 such that $C_2 - A_2 \geq 0$. Using the scheme of (4.4) and (4.5) with B_κ^2 replaced by $C_2 - A_2$ (and using (4.9) of course) we obtain the estimate

$$\langle \text{Op}^w(e) \rangle_u \leq C_1 \|u\|_{-1}^2 + \sigma \|u\|_{-t}^2 + C_2 \sigma^{-1} \|v\|_t^2; \quad \sigma > 0. \quad (4.10)$$

4.3. Bounds at \mathcal{F}^+ . Let $z^+(\cdot) \in \mathcal{F}^+$. We use the following construction, cf. (4.1), with l defined in terms of z^+ :

$$b = X^{1/2}a; \quad a = X^{-t}F(r > 1)f(h)F(l < \epsilon).$$

Mimicking the proof of (4.7) with the quantization of this symbol and using (4.10) we derive the bound

$$\begin{aligned} \|\tilde{A}u\|_{-t}^2 &\leq C_1 \|u\|_{-1}^2 + C_2 \|v\|_t^2 + \sigma \|u\|_{-t}^2; \\ \tilde{A} &= \text{Op}^w(F(r > 1)f(h)F(l < \epsilon)), \quad \sigma > 0, \quad C_j = C_j(\sigma). \end{aligned} \quad (4.11)$$

5. BOUNDS AT $\mathcal{F}_{\text{sa}}^-$

In this and the following section we shall need Conditions 2.1 and 2.4. Let u and v be given as in Section 4. We shall prove

$$\begin{aligned} \|\tilde{A}u\|_{t-1}^2 &\leq C_1 \|u\|_{-1}^2 + C_2 \|v\|_t^2; \\ \tilde{A} &= \text{Op}^w(F(r > 1)f(h)F(l < \epsilon)), \end{aligned} \quad (5.1)$$

where $l = l_h$ is the function associated to a given branch $z^-(\cdot) \in \mathcal{F}_{\text{sa}}^-$, cf. (2.14).

Using the observable a_2 of (2.20) we introduce the following equivalence relation on the set $\mathcal{F}_{\text{sa}}^-(E)$: $z \sim \tilde{z}$ for $z, \tilde{z} \in \mathcal{F}_{\text{sa}}^-(E)$ if and only if $a_2(z) = a_2(\tilde{z})$. On the set $\mathcal{F}_{\text{sa}}^-(E)/\sim$ we have the following partial ordering: $z \prec \tilde{z}$ for $z, \tilde{z} \in \mathcal{F}_{\text{sa}}^-(E)$ if and only if $a_2(z) \leq a_2(\tilde{z})$.

We have the important property that for $z \prec \tilde{z}$ any incoming orbit at the representative z can not emanate from the representative \tilde{z} . In particular if $z(E) \in \mathcal{F}_{\text{sa}}^-(E)$ belongs to a minimal class then by Condition 2.4 the two incoming orbits at the representative $z(E)$ will come from $\mathcal{F}_{\text{so}}^-(E)$, cf. Proposition 2.3 and Condition 2.4.

Suppose first that $z(E) \in \mathcal{F}_{\text{sa}}^-(E)$ belongs to a minimal class at a fixed energy $E = E_0 \in I$. As a preliminary step we are interested in quantizing the symbols (4.1) with f supported in an interval $[E_0 - \delta, E_0 + \delta]$ with $\delta \ll \epsilon$. The unstable manifold at neighboring E 's is denoted by M_E^u . We introduce $\mathbb{T}_-^* = \{z \in \mathbb{T}^* \mid \rho < 0\}$, and define for $\epsilon > 0$

$$K = \left\{ z \in \mathbb{T}_-^* \mid \frac{\epsilon}{4} \leq l_{E_0} \leq 2\epsilon, h = E_0, \text{dist}(z, M_{E_0}^u) \geq \frac{\epsilon}{2} \right\}.$$

If ϵ is small enough, then for all $z \in K$ we have $\phi_\tau(z) \rightarrow \tilde{z} \in \mathcal{F}_{\text{so}}^-$ for $\tau \rightarrow -\infty$. This means that $\phi_\tau(z)$ in the far past will belong to a set where we have a good wave front set bound due to the result of Subsection 4.1. Since K is compact we may choose $\tau_0 \ll -1$ such that we have a good wave front set bound of an open neighborhood $U_{\tau_0} \subseteq \mathbb{T}^*$ of the compact set $\phi_{\tau_0}(K)$. Define $U_1 = \phi_{-\tau_0}(U_{\tau_0})$. By Proposition 3.3 there are good bounds on compact subsets of U_1 . Fix such $\epsilon > 0$ and such open set U_1 .

Next we define

$$U_2^\delta = \{z \in \mathbb{T}_-^* \mid \frac{\epsilon}{4} < l_h < 2\epsilon, |h - E_0| < 2\delta, \text{dist}(z, M_h^u) < \epsilon\}; \delta > 0,$$

and also

$$K^\delta = \{z \in \mathbb{T}_-^* \mid \frac{\epsilon}{2} \leq l_h \leq \epsilon, |h - E_0| \leq \delta, \text{dist}(z, M_h^u) \geq \epsilon\}; \delta \geq 0.$$

Lemma 5.1. *For all $\delta > 0$ small enough, $K^\delta \subseteq U_1$.*

Proof. Clearly $K^0 \subseteq U_1$. Using that $\nabla h(z(E_0)) \neq 0$ and the fact that $|h(z) - E_0| \leq \delta$ for any given $z \in K^\delta$ we may choose a neighboring \tilde{z} with $h(\tilde{z}) = E_0$; $|z - \tilde{z}| \leq C\delta$. Assuming $\delta \ll \epsilon$ we see that $\tilde{z} \in K$. We may assume that $\text{dist}(K, \mathbb{T}^* \setminus U_1) > C\delta$. Whence $z \in U_1$. \square

Due to the lemma we may choose $\delta > 0$ such that

$$\tilde{K} := \{z \in \mathbb{T}_-^* \mid \frac{\epsilon}{2} \leq l_h \leq \epsilon, |h - E_0| \leq \delta\} \subseteq U_1 \cup U_2^\delta.$$

Next pick non-negative $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^*)$ subordinate to this covering, that is $\text{supp } \psi_1 \subseteq U_1$, $\text{supp } \psi_2 \subseteq U_2^\delta$ and $\psi_1 + \psi_2 = 1$ on \tilde{K} .

For any f supported in an interval $[E_0 - \delta, E_0 + \delta]$ we may decompose $F'(l_h < \epsilon)f(h) = (\psi_1 + \psi_2)F'(l_h < \epsilon)f(h)$. In agreement with the discussion above the contribution from the term involving ψ_1 may be treated by (4.7) and Propositions 2.3 and 3.3 yielding a bound of the type (4.7). The contribution from the term involving ψ_2 has the right sign (more precisely, $\psi_2\{h, l_h\} \geq 0$). We conclude (5.1) for z^- for which $a_2(z^-(E_0))$ is minimal under the additional condition that f is supported near the fixed $E_0 \in I$.

By repeating these arguments for the other $z^- \in \mathcal{F}_{\text{sa}}^-$ in the order of increasing a_2 , measured at $E = E_0$, we conclude (5.1) for all $z^- \in \mathcal{F}_{\text{sa}}^-$.

Now by letting $E_0 \in I$ vary, the bound for any $f \in C_0^\infty(I)$ finally follows by a compactness argument.

6. LIMITING ABSORPTION PRINCIPLE

Let us define $u = R(\zeta)v = (H - \zeta)^{-1}v$ for $v \in L^{2,t}$, $t \in (1/2, 1)$, and ζ with $\Re\zeta \in [a, b] \subseteq I$ and $\Im\zeta > 0$. We aim at the following limiting absorption principle

bound,

$$\|u\|_{-t} \leq C\|v\|_t. \quad (6.1)$$

Suppose (6.1) is not true. Then

$$\|u_n\|_{-t} > n\|v_n\|_t \quad (6.2)$$

for sequences (u_n) and (v_n) of functions of this type and with $\zeta_n \rightarrow E \in I$. We normalize u_n in $L^{2,-t}$. The resulting new sequence (whose elements are also denoted by u_n) has a weak limit, say u , in $L^{2,-t}$. Clearly $(H - E)u = 0$ in the distributional sense.

6.1. u not zero. We combine the bounds (4.7), (4.10), (4.11) and (5.1) to one estimate

$$\|u_n\|_{-t}^2 \leq C_1\|u_n\|_{-1}^2 + C_2\|(H - \zeta_n)u_n\|_t^2. \quad (6.3)$$

Letting $n \rightarrow \infty$ using a compactness argument yields

$$1 \leq C_1\|u\|_{-1}^2.$$

Clearly it follows that

$$u \neq 0 \text{ in } L^{2,-1}. \quad (6.4)$$

Remark 6.1. One may also derive (6.3) using the observable

$$a_3 = C(g'_\epsilon \rho - \tilde{b}) - 2(b\rho + V')\eta; \quad g_\epsilon(r) = r - r^{1-\epsilon}, \quad \epsilon > 0.$$

This is done by showing that

$$\{h, F(r > 1)f(h)a_3\} \geq \sigma r^{-1-\epsilon}F(r > 1)f(h) + O(r^{-1-2\epsilon})$$

for $F(r > 1)f(h)$ as in (4.1) and for a small $\sigma > 0$; and similarly for a quantization, cf. Subsection 4.2.

Here Condition 2.4 is not used. However we will need Condition 2.4 for the bound (6.5) stated below.

6.2. Weak decay. We shall show that

$$u \in B_0^*. \quad (6.5)$$

First we apply (4.7), (4.10) and (5.1) to u_n and let $n \rightarrow \infty$. The result reads

$$WF^{t-1}(u) \cap \mathcal{F}^- = \emptyset, \quad (6.6)$$

and

$$WF^{-\frac{1}{2}}(u) \subseteq \mathcal{F}. \quad (6.7)$$

In particular

$$WF^{-\frac{1}{2}}(u) \subseteq \mathcal{F}^+,$$

which in conjunction with Proposition 3.5 yields (6.5).

6.3. Strong decay. In this section we shall improve (6.5) along the line of the proof of [Hö1, Theorem 30.2.9].

Proposition 6.2. *The function u is in $L^{2,1}$ and obeys*

$$\|u\|_1 \leq C\|u\|_{-1}, \quad (6.8)$$

where C is independent of $E \in [a, b]$.

Proof. First we shall prove that

$$WF^{t-1}(u) = \emptyset. \quad (6.9)$$

For any $z^+ \in \mathcal{F}_{\text{si}}^+$ we may use the construction (4.1) with l defined in terms of z^+ to show that indeed

$$z^+ \notin WF^{t-1}(u). \quad (6.10)$$

In fact we may follow the scheme of Subsection 4.1 with ζ , u and $v = (H - \zeta)u$ replaced by $E \in I$, $u_R = \psi_R u$ and $v_R = (H - E)u_R$, respectively. Here $\psi_R(x) = \psi(r/R)$ is given as in the proof of Proposition 3.5. Notice that for this choice of b_κ the Poisson bracket is now non-negative (this would not allow us to have a complex ζ). Using the estimate (6.5) allows us to take $R \rightarrow \infty$ in the commutator of (4.5)

$$\lim_{R \rightarrow \infty} \langle i[H - E, B_\kappa^2] \rangle_{u_R} = 0.$$

Consequently (4.4) leads to the following bound (given in terms of the new l)

$$\|X_\kappa^{-1} A_\kappa u\|^2 \leq C \|u\|_{-1}^2,$$

from which we in turn obtain (6.10) by letting $\kappa \rightarrow 0$.

Thus

$$WF^{t-1}(u) \cap \mathcal{F}_{\text{si}}^+ = \emptyset. \quad (6.11)$$

Next we repeat the above arguments as well as those of Section 5 for the fixed points $z^+ \in \mathcal{F}_{\text{sa}}^+$ now arranged in the order of *decreasing* a_2 ; here we use the bound (6.11) and Proposition 3.1 (for the state u) in the backward direction of the flow, cf. Remark 3.4. This leads to

$$WF^{t-1}(u) \cap \mathcal{F}_{\text{sa}}^+ = \emptyset. \quad (6.12)$$

In combination with Proposition 3.1 the estimates (6.6), (6.11) and (6.12) yield (6.9).

Next we notice that the above regularization procedure in conjunction with (6.9) and Section 5 may work to improve (6.6) by one power, $t - 1 \rightarrow t$. The arguments above yield an improvement of (6.11) and (6.12) by one power. Whence we obtain (6.9) with $t - 1 \rightarrow t$. By bootstrapping we conclude that in fact $u \in L^{2,\infty}$, in particular we infer that $u \in L^{2,1}$.

The estimate (6.8) follows by keeping track of bounding constants, cf. Remark 3.4. \square

Remark 6.3. It follows from the above proof that in fact Proposition 6.2 holds for all $u \in B_0^*$ such that $(H - E)u = 0$ with $E \in [a, b]$.

6.4. Completing the proof of LAP. In this section we complete the proof of limiting absorption principle. We mimic the proof of [Hö1, Theorem 30.2.10].

It follows from (6.4) and Proposition 6.2 that u is an L^2 -eigenfunction, and from the uniformity in $E \in [a, b]$ and a compactness argument that the set of E 's constructed as above must be finite in $[a, b]$. Thus for $\Re \zeta$ staying away from a finite set (by a positive distance) we deduce (6.1) which in turn implies absence of eigenvalues. Thus we have shown that $\sigma_{pp}(H)$ is discrete in I . (Alternatively we may invoke Remark 6.3 to show this.)

If $[a, b]$ does not contain eigenvalues of H the bound (6.2) lead to a contradiction and therefore (6.1) holds with a constant independent of ζ . We have shown:

Theorem 6.4. *With the assumptions from Section 2 (including Conditions 2.1 and 2.4) $\sigma_{pp}(H)$ is discrete in I . Moreover for any subinterval $[a, b] \subset I \setminus \sigma_{pp}(H)$ and $t > 1/2$ there is a constant $C > 0$ such that*

$$\|R(E + i\epsilon)\|_{\mathcal{B}(L^{2,t}, L^{2,-t})} \leq C; \quad E \in [a, b], \quad \epsilon > 0. \quad (6.13)$$

Corollary 6.5. *Suppose the conditions of Theorem 6.4 and*

$$I \cap \sigma_{pp}(H) = \emptyset. \quad (6.14)$$

Then for all $s > 2^{-1}$ and $f \in C_0^\infty(I)$ there exists a constant $C > 0$ such that

$$\int_{-\infty}^{\infty} \|X^{-s} e^{-itH} f(H)\psi\|^2 dt \leq C\|\psi\|^2. \quad (6.15)$$

Remark 6.6. We introduce a class of perturbations, cf. (2.1), for which small modifications of the methods of this paper may yield generalizations of our results. In this sense it is not an “optimal” class of perturbations, but for convenience let us assume:

The scalar potential $V_\delta = V_\delta^1 + V^2$ where V^2 is real-valued, compactly supported and Laplacian bounded with norm less than 1. The potentials \mathbf{a}_δ and V_δ^1 are smooth (with values in \mathbb{R}^2 and \mathbb{R} , respectively) and obey the bounds

$$\partial_x^\alpha \mathbf{a}_\delta = \mathcal{O}(|x|^{-\delta-|\alpha|}) \text{ and } \partial_x^\alpha V_\delta^1 = \mathcal{O}(|x|^{-\delta-|\alpha|}), \quad (6.16)$$

for some $\delta > 0$.

Upon adding such perturbations to the expression (3.3), $\delta > 0$ suffices for Sections 3–8. For Sections 9–11 and Section 13 one would need $\delta > 1$.

7. PRELIMINARY ESTIMATES

We impose in this and in the following two sections Conditions 2.1 and 2.4. As in the previous sections I refers to an open interval obeying (2.4). We shall also assume (6.14) in this and the following sections. Our bounds involve functions $f, \tilde{f} \in C_0^\infty(I)$ such that $0 \leq \tilde{f} \leq 1$ and $\tilde{f} = 1$ in a neighborhood of $\text{supp } f$. Moreover e refers to the symbol (4.8) with f replaced by \tilde{f} , and the subscript t indicates “expectation” in the state

$$\psi(t) = e^{-itH} f(H)\psi; \text{ viz. } \langle A \rangle_t := \langle \psi(t), A\psi(t) \rangle.$$

For convenience let us in the following assume that there exists $\epsilon_0 > 0$ such that

$$V(\theta) \in I \Rightarrow |V'(\theta)| \geq 2\epsilon_0, \quad (7.1)$$

cf. (2.4). (Obviously this may be achieved possibly by a slight shrinking of I .) Let $\epsilon_1 > 0$ obey

$$2\epsilon_1 \max |b| \leq \epsilon_0, \quad 2\epsilon_1^2 \leq \text{dist}(\text{supp } \tilde{f}, \mathbb{R} \setminus I). \quad (7.2)$$

Lemma 7.1. *For all functions f, \tilde{f} , constants $\epsilon_1 > 0$ and states $\psi(t)$ as above*

$$\int_1^\infty |\langle \text{Op}^w(e) \rangle_t| dt \leq C\|\psi\|^2, \quad (7.3)$$

$$\int_1^\infty t^{-1} \langle F'(\frac{r}{t} > \bar{C}) \rangle_t dt \leq C\|\psi\|^2; \quad (7.4)$$

$$\bar{C} > 2\sqrt{2(\text{sup}(\text{supp } \tilde{f}) - \text{min } V)},$$

$$\int_1^\infty t^{-1} |\langle \text{Op}^w(b_1) \rangle_t| dt \leq C\|\psi\|^2; \quad (7.5)$$

$$b_1 = F(r > 1)\tilde{f}(h)F(\frac{r}{t} < \bar{C})F(\rho < \epsilon_1),$$

$$\int_1^\infty t^{-1} |\langle \text{Op}^w(b_2) \rangle_t| dt \leq C\|\psi\|^2; \quad (7.6)$$

$$b_2 = F(r > 1)\tilde{f}(h)F(\frac{r}{t} < \frac{\epsilon_1}{2})F(\rho > \epsilon_1).$$

Proof. The bound (7.3) follows from (4.9) and Corollary 6.5.

The maximal velocity bound (7.4) is rather standard, see for example [CHS, Lemma 6.4] or [HS2, Lemma 8.1].

As for the bound (7.5) we first show that

$$\int_1^\infty t^{-1} |\langle \text{Op}^w(c_1) \rangle_t| dt \leq C \|\psi\|^2, \quad (7.7)$$

where

$$c_1 = F(r > 1) \tilde{f}(h) F\left(\frac{r}{t} < \bar{C}\right) F(|\rho| < 2\epsilon_1).$$

For that we notice that for $\epsilon' > 0$ small enough

$$F(r > 1) \tilde{f}(h) F(|\rho| < 2\epsilon_1) F((b\rho + V')^2 + \eta^2 < \epsilon') = 0. \quad (7.8)$$

Here we used (7.1) and (7.2).

We infer (7.7) from Corollary 6.5, (7.3) and (7.8).

To show (7.5) we introduce the family of “propagation observables”

$$\Phi(t) = \text{Op}^w(c_2); \quad c_2 = \frac{r}{t} b_1.$$

We consider the expectation of the Heisenberg derivative $\mathbf{D}\Phi(t) = i[H, \Phi(t)] + \frac{d}{dt}\Phi(t)$ in the state $\psi(t)$. We shall henceforth denote by $\mathbf{d}a = \{h, a\} + \frac{d}{dt}a$ the classical Heisenberg derivative of an observable a . For the first factor we may use that

$$\mathbf{d}(t^{-1}r) = t^{-1}\left(\rho - \frac{r}{t}\right). \quad (7.9)$$

Clearly

$$\begin{aligned} & \left(\rho - \frac{r}{t}\right) F(\rho < \epsilon_1) (1 - F(|\rho| < 2\epsilon_1)) \\ & \leq -\epsilon_1 F(\rho < \epsilon_1) (1 - F(|\rho| < 2\epsilon_1)). \end{aligned} \quad (7.10)$$

We obtain (7.5) using Corollary 6.5, (7.4), (7.7), (7.9), (7.10) and the fact that $\Phi(t)$ is uniformly bounded.

The bound (7.6) follows similarly, cf. [HS2, Lemma 8.1]. \square

Lemma 7.2. *With functions f, \tilde{f} , constants $\epsilon_1, \bar{C} > 0$, states $\psi(t)$ and symbols b_1, b_2 as in Lemma 7.1*

$$\lim_{t \rightarrow \infty} \|F\left(\frac{r}{t} > \bar{C}\right) \psi(t)\| = 0, \quad (7.11)$$

$$\lim_{t \rightarrow \infty} \langle \text{Op}^w(b_1) \rangle_t = 0, \quad (7.12)$$

$$\lim_{t \rightarrow \infty} \langle \text{Op}^w(b_2) \rangle_t = 0, \quad (7.13)$$

$$\lim_{t \rightarrow \infty} \langle \text{Op}^w(b_3) \rangle_t = 0; \quad (7.14)$$

$$b_3 = re = F(r > 1) \tilde{f}(h) ((b\rho + V')^2 + \eta^2).$$

Proof. The maximal velocity bound (7.11) is standard, see for example [CHS, Proposition 6.8]. The bound (7.12) follows from (7.5) and the fact that $\frac{d}{dt} \langle \text{Op}^w(b_1) \rangle_t$ is integrable, cf. the proof of Lemma 7.1. We argue similarly for (7.13). For (7.14) we may proceed as follows: By (7.3) and (7.11) it suffices to show that the time-derivative of $\langle \text{Op}^w(b_3) \rangle_t$ is integrable. For that we use Corollary 6.5, (2.19) and (7.3). \square

8. PROJECTIONS P_j

Motivated by Lemma 7.2 we shall for all branches $z_j = z_j(E) \in \mathcal{F}^+$, $E \in I$, associate a projection $P_j : \text{Ran}(1_I(H)) \rightarrow \text{Ran}(1_I(H))$. It is characterized in terms of the strong limit

$$P_j f(H) = s - \lim_{t \rightarrow \infty} e^{itH} \text{Op}^w(\chi_j \tilde{f}) e^{-itH} f(H). \quad (8.1)$$

Here $f \in C_0^\infty(I)$ is arbitrary and the symbol $\chi_j \tilde{f}$ is given in terms of any function $\tilde{f} \in C_0^\infty(I)$ such that $0 \leq \tilde{f} \leq 1$ and $\tilde{f} = 1$ in a neighborhood of $\text{supp } f$, and in terms of any small $\epsilon > 0$, as

$$\chi_j \tilde{f} = F(r > 1) \tilde{f}(h) F(|\theta - \theta_j(h)| < \epsilon). \quad (8.2)$$

(We have suppressed a trivial periodization in the variable θ .)

Proposition 8.1. *Given (6.14), the expression (8.1) is well-defined and independent of the choices of \tilde{f} and small ϵ in (8.2). Taking $f \uparrow 1_I$ the resulting limit $P_j = s - \lim_{f \uparrow 1_I} P_j f(H)$ is indeed an orthogonal projection with $\text{Ran}(P_j) \subseteq \text{Ran}(1_I(H))$. Moreover, P_j reduces H .*

Proof. We introduce an “intermediate” f_1 , that is $f_1 = 1$ in a neighborhood of $\text{supp } f$ and $\tilde{f} = 1$ in a neighborhood of $\text{supp } f_1$. To show the existence of the limit on the right hand side of (8.1) it suffices by commutation, Corollary 6.5 and Lemma 7.2 to prove the existence of

$$P_{j,f} = s - \lim_{t \rightarrow \infty} f_1(H) e^{itH} \text{Op}^w\left(\chi_j \tilde{f} F\left(\frac{r}{t} > \frac{\epsilon_1}{4}\right) F\left(\frac{r}{t} < \bar{C}\right)\right) e^{-itH} f(H).$$

For that we differentiate and need to verify integrability. Clearly the contribution to the Heisenberg derivative from the factor $F\left(\frac{r}{t} > \frac{\epsilon_1}{4}\right) F\left(\frac{r}{t} < \bar{C}\right)$ may be treated by (7.4)–(7.6).

The Poisson bracket with the factor $F(|\theta - \theta_j(h)| < \epsilon)$ is obviously proportional to $r^{-1} F'(|\theta - \theta_j(h)| < \epsilon)$. This term is treated as follows: Let $\epsilon_1 > 0$ be given as in Lemma 7.1. Then for a sufficiently small $\epsilon' > 0$ (in particular $\epsilon' \ll \epsilon$)

$$\begin{aligned} F'(|\theta - \theta_j(h)| < \epsilon) &= F'(\cdot) F(\rho < \epsilon_1) \\ &\quad + F'(\cdot) F(\rho > \epsilon_1) F(\eta^2 + (b\rho + V')^2 > \epsilon'). \end{aligned} \quad (8.3)$$

Here we used (2.9).

The contribution from the first term on the right hand side of (8.3) is treated using (7.5); the second term by (7.3).

We conclude that the limit on the right hand side of (8.1) exists. Clearly $P_{j,f}$ is independent of the choices of f_1 , \tilde{f} and ϵ , cf. (8.3). In particular there is a unique orthogonal projection P_j (given by extension by continuity using the spectral theorem) such that

$$\begin{cases} P_j \psi = 0 & \text{for } \psi \in \text{Ran}(1_{\mathbb{R} \setminus I}(H)) \\ P_j f(H) \psi = P_{j,f} \psi & \text{for } f \in C_0^\infty(I) \text{ and for arbitrary } \psi \end{cases}.$$

Notice the formula $P_{j,g} f(H) = P_{j,f}$ which holds when $g = 1$ on the support of f for $f, g \in C_0^\infty(I)$. This shows that $P_j f(H) \psi = P_{j,g} f(H) \psi$ and hence in particular that P_j is well-defined. Obviously P_j reduces H . \square

It follows from Lemma 7.2 and Proposition 8.1 that the projections span the spectral subspace for the interval I .

Proposition 8.2. *Given (6.14),*

$$1_I(H) = \sum_j P_j. \quad (8.4)$$

We may supplement the list of “global” estimates in Lemma 7.2 by the following ones for states associated to P_j . Let

$$\rho_j(E) = \sqrt{2(E - V(\theta_j(E)))}; \quad E \in I. \quad (8.5)$$

Lemma 8.3. *For all small $\bar{\epsilon} > 0$, functions f, \tilde{f} as above and $\psi = f(H)\psi \in \text{Ran}(P_j)$*

$$\|\text{Op}^w(d_1)\psi(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty; \quad (8.6)$$

$$d_1 = F(r > 1)\tilde{f}(h)F(|\rho - \rho_j(h)| > \bar{\epsilon}),$$

$$\|\text{Op}^w(d_2)\psi(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty; \quad (8.7)$$

$$d_2 = F(r > 1)\tilde{f}(h)F\left(\left|\frac{r}{t} - \rho_j(h)\right| > \bar{\epsilon}\right).$$

Proof. The bound (8.6) follows from the energy relation $\rho^2 + \eta^2 = 2(h - V)$ and Lemma 7.2.

As for the bound (8.7) we introduce the observable $\Phi(t) = \text{Op}^w(c_1)$ with

$$\begin{aligned} c_1 = & F(r > 1)\tilde{f}(h)F(|\theta - \theta_j(h)| < \epsilon) \times \\ & F\left(\frac{r}{t} > \frac{\epsilon_1}{4}\right)F\left(\frac{r}{t} < \bar{C}\right)F(|\rho - \rho_j(h)| < \epsilon')F((b\rho + V')^2 + \eta^2 < \epsilon'') \times \\ & \left|\frac{r}{t} - \rho_j(h)\right|F\left(\left|\frac{r}{t} - \rho_j(h)\right| > \bar{\epsilon}\right). \end{aligned}$$

We choose $0 < \epsilon, \epsilon'' \ll \epsilon'$. Then the contribution to the expectation of the Heisenberg derivative of $\Phi(t)$ in the state $\psi(t)$ from differentiating the factor $F(|\rho - \rho_j(h)| < \epsilon')$ vanishes (by the energy relation). As for the contribution from the last factor we write, cf. (7.9),

$$\mathbf{d}(t^{-1}r) = t^{-1}\left((\rho - \rho_j(h)) - \left(\frac{r}{t} - \rho_j(h)\right)\right). \quad (8.8)$$

The first term on the right hand side of (8.8) is bounded by ϵ' . Hence (given that $2\epsilon'$ is smaller than $\bar{\epsilon}$) the second term “dominates”. This fact and previously used estimates yields an integral estimate for a symbol c_2 given by omitting the outer factor $\left|\frac{r}{t} - \rho_j(h)\right|$ in the expression c_1 . By the standard method, see the proof of (7.12), we readily obtain from this bound that indeed (8.7) holds. \square

9. $P_j = 0$ AT $\mathcal{F}_{\text{sa}}^+$

In this section we shall invoke [HS2] to show that there are no states associated to any branch $z_j = z_j(\cdot) \in \mathcal{F}_{\text{sa}}^+$.

As in [HS1] and [HS2] we linearize the equations of motion nearby any $z_j(\cdot) \in \mathcal{F}^+$ as follows: Write for $E \in I$

$$\omega_1(E) = (\cos \theta_j(E), \sin \theta_j(E)), \quad \omega_2(E) = (-\sin \theta_j(E), \cos \theta_j(E)),$$

and introduce new coordinates

$$x = x_1(\omega_1(E) + u\omega_2(E)), \quad \xi = \xi(E) + \mu\omega_1(E) + v\omega_2(E),$$

where (with $\rho(E) = \rho_j(E)$ given by (8.5))

$$\xi(E) = \rho(E)\omega_1(E) + b(\theta_j(E))\omega_2(E).$$

Notice that

$$x_1 = x \cdot \omega_1(E) \text{ and } u = \frac{x_2}{x_1}; \quad x_2 = x \cdot \omega_2(E). \quad (9.1)$$

We may express μ as a function of u, v and E (for small u and v) using the energy relation. Introducing the “time” $\bar{\tau} = \ln x_1(t)$ we obtain an autonomous

system whose linearization at the fixed point $(u, v) = 0$ takes the following form: For $w = (u, v)^{\text{tr}}$

$$\frac{d}{d\bar{r}}w = Bw; \quad B = \rho^{-1} \begin{pmatrix} -b' - \rho & 1 \\ -d & b' \end{pmatrix}; \quad d = b^2 + b'^2 + 2b'\rho + V''.$$

Here B is evaluated at $(E, \theta_j(E))$. We used [HS2, (1.7)] and the leading asymptotics

$$\mu = -\rho^{-1} \left(\frac{1}{2}du^2 - b'vu + \frac{1}{2}v^2 \right) + \mathcal{O}(|w|^3).$$

The eigenvalues of B are given by dividing those of (2.11) by ρ , that is

$$\beta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\frac{\kappa^+}{\rho^2}}, \quad \tilde{\beta} = -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\frac{\kappa^+}{\rho^2}}. \quad (9.2)$$

To apply [HS2] we need to verify various conditions. As for the non-resonance condition [HS2, (H8)] we consider

Condition 9.1 (Few resonances condition). At the branch $z_j(\cdot) \in \mathcal{F}^+$ there is a discrete set $\mathcal{D} \subset I$ such that

$$\frac{d}{dE} \left(\frac{\kappa^+}{\rho^2} \right) \neq 0 \text{ for } E \in I \setminus \mathcal{D}. \quad (9.3)$$

With Condition 9.1 imposed for a branch $z_j(\cdot) \in \mathcal{F}^+$ it is easy to see that the resonances of fixed order can accumulate at most at a discrete set of energies in I . We recall that $E \in I$ is resonance of order $m \geq 2$ if we may write

$$\beta^\natural = m_1\beta + m_2\tilde{\beta}, \quad (9.4)$$

where β^\natural signifies either β or $\tilde{\beta}$ and the coefficients to the right are non-negative integers with $m_1 + m_2 = m$.

For the case of $z_j(\cdot) \in \mathcal{F}_{\text{sa}}^+$ that we address in this section it is enough to avoid (9.4) with only $\beta^\natural = \tilde{\beta}$ to the left and for all orders $m \leq m_0$, where m_0 is sufficiently large, see [HS2, (H8)]. Under Condition 9.1 the “bad” subset of I (where (9.4) holds for some $m \leq m_0$) can be “reached” using the spectral theorem. (Here we use the fact that its set of accumulation points is at most discrete.)

For completeness of presentation we notice that in the case of $z_j(\cdot) \in \mathcal{F}_{\text{sa}}^+$ the union of resonances of all orders is dense.

Since the other conditions [HS2, (H1)–(H7)] are readily verified using Lemmas 7.1, 7.2 and 8.3 we conclude from [HS2, Theorem 1.1]:

Theorem 9.2. *Suppose Condition 9.1 for a branch $z_j(\cdot) \in \mathcal{F}_{\text{sa}}^+$. Then*

$$P_j 1_I(H) = 0. \quad (9.5)$$

We notice that Condition 9.1 is trivially fulfilled in the case $V = 0$.

10. WAVE OPERATORS AT $\mathcal{F}_{\text{si}}^+$

In this section we shall consider two types of wave operators at any branch $z_j(\cdot) \in \mathcal{F}_{\text{si}}^+$. We construct one for high energies, cf. [HS1, Theorem 1.2], and with a further assumption one for small energies, cf. [HS1, Theorem 1.1]. With this extra assumption there will be an interval of overlap, so that for this energy regime both wave operators exist. The constructions do not need Conditions 2.1, 2.4 nor (6.14).

10.1. Wave operator at high energies. We assume that for all $E \in I$

$$0 < 4\kappa^+ < \rho^2 \text{ at } z_j(E); \quad (10.1)$$

or alternatively that both eigenvalues in (9.2) are negative reals throughout $E \in I$.

Let us look at the initial value problem

$$\frac{d\eta}{d\theta} = -\frac{1}{\eta} \left\{ (\eta + b) \sqrt{2(E - V) - \eta^2 + V'} \right\}, \quad \eta(\theta_j) = 0, \quad (10.2)$$

nearby θ_j where $z_j(\cdot) = (\theta_j(\cdot), 0, \rho_j(\cdot)) \in \mathcal{F}_{\text{si}}^+$. Formally the differential equation arises from eliminating ρ and θ in the system (2.6). Obviously it is a *singular* ordinary differential equation. We can solve it away from resonances as follows.

Let us denote by λ the biggest of the two eigenvalues in (2.11) and $\tilde{\lambda}$ the smallest one (cf. the notation (9.2)). Away from resonances the Sternberg diffeomorphism provides integral curves for the system (2.6) (with ρ eliminated) of the form

$$(\theta(\tau) - \theta_j, \eta(\tau)) = \Psi(+(-)e^{\lambda^{\natural}\tau}(1, \lambda^{\natural})); \quad \tau \text{ large}. \quad (10.3)$$

We use (10.3) with $\lambda^{\natural} = \lambda$ and look at the parametrized curve

$$(\theta - \theta_j, \eta) = \Psi(s(1, \lambda)); \quad |s| \text{ small.}$$

Using θ as a new parameter we obtain a solution $\eta = \eta_E(\theta)$ to (10.2) with

$$\eta = \lambda(\theta - \theta_j) + \mathcal{O}(|\theta - \theta_j|^2). \quad (10.4)$$

We notice that this η is smooth as a function of (E, θ) (away from resonances), see [HS1, Appendix]. Also we remark that although it is not unique as a solution to the initial value problem (10.2) and subject to (10.4) (there is a one-parameter family of C^1 -solutions), all derivatives $\eta_E^{(k)}(\theta_j)$ are uniquely determined by these requirements; for this assertion E needs to be non-resonance. Following [CHS, Subsection 3.5] we introduce

$$S_E = r\rho_E; \quad \rho_E = \rho_E(\theta) = \sqrt{2(E - V) - \eta_E^2}. \quad (10.5)$$

This function, $S_E = S_E(r, \theta) = S_E(x)$, solves the eikonal equation

$$h(x, \nabla_x S_E) = 0.$$

Using the formulas

$$\partial_E \rho_E = \rho_E^{-1}, \quad \partial_E^2 \rho_E = -(1 + (\rho_E \partial_E \eta_E)^2) \rho_E^{-3}; \quad \text{at } \theta = \theta_j, \quad (10.6)$$

we may use a Legendre transformation to obtain a smooth solution $S = S(t, r, \theta) = S(t, x)$ for $t > 0$ to the Hamilton–Jacobi equation

$$h(x, \nabla_x S) = -\partial_t S,$$

cf. [CHS]. Assuming that $I = (E^-, E^+)$ is small centered around, say E^0 , and does not contain resonances, cf. Condition 9.1, S is defined in a region $\mathcal{D}_\epsilon = \cup_{t>0} \{t\} \times \mathcal{D}_{\epsilon,t}$ where $x \in \mathcal{D}_{\epsilon,t}$ (with $\epsilon > 0$ taken small) if and only if

$$|\theta - \theta_j(E^0)| < \epsilon, \quad (\partial_E \rho)^{-1}(E = E^-, \theta) < \frac{r}{t} < (\partial_E \rho)^{-1}(E = E^+, \theta).$$

It is given explicitly as $S = r\rho_E - tE$ where $E \in I$ is determined by

$$\frac{r}{t} = (\partial_E \rho_E)^{-1}(\theta), \quad (10.7)$$

and possesses the homogeneity property

$$S(t, r, \theta) = tS(1, \frac{r}{t}, \theta).$$

Following [CHS] we define the “direct” flow

$$\begin{cases} \frac{d\tilde{\theta}}{dt}(t) = \tilde{r}^{-1}[\tilde{r}^{-1}\partial_{\theta}S(t, \tilde{r}, \tilde{\theta}) - b(\tilde{\theta})] \\ \frac{d\tilde{r}}{dt}(t) = \partial_r S(t, \tilde{r}, \tilde{\theta}) \end{cases} . \quad (10.8)$$

The domain \mathcal{D}_ϵ is preserved under the forward direct flow, and there is energy conservation

$$-\partial_t S(t, \tilde{r}(t), \tilde{\theta}(t)) = \text{const},$$

cf. [CHS, Lemma 3.11].

Using the “time” τ of (2.5) we may rewrite (10.8) as the following system of equations

$$\begin{cases} \frac{d\tilde{\theta}}{d\tau}(\tau) = \eta_E(\tilde{\theta}) \\ \frac{d\tilde{r}}{d\tau}(\tau) = \tilde{r}\rho_E(\tilde{\theta}) \end{cases} . \quad (10.9)$$

Let us find the asymptotics of the quantity $(\tilde{\theta} - \theta_j)\tilde{r}^{-\beta}$ (with β given as in (9.2)) for the solution to (10.9) starting at time $\tau = 0$ at $(\tilde{\theta}_0, \tilde{r}_0)$. By integrating the first equation in (10.9) we obtain

$$\tilde{\theta}(\tau) - \theta_j = (\tilde{\theta}_0 - \theta_j)e^{\lambda\tau}(1 + o(1)). \quad (10.10)$$

Clearly the solution to the second equation is

$$\tilde{r}(\tau) = \tilde{r}_0 e^{\int_0^\tau \rho_E(\tilde{\theta}(\tau')) d\tau'}. \quad (10.11)$$

We write

$$\rho_E(\theta) = \rho_E(\theta_j)(1 + \zeta_E(\theta)\eta_E(\theta)), \quad (10.12)$$

where we have introduced the smooth function

$$\zeta_E(\theta) = \frac{\sqrt{1 + \frac{2V(\theta_j) - 2V(\theta) - \eta_E(\theta)^2}{\rho_E(\theta_j)^2}} - 1}{\eta_E(\theta)}.$$

Using (10.12) and the first equation of (10.9) we may write (10.11) as

$$\tilde{r}(\tau) = \tilde{r}_0 e^{\rho_E(\theta_j)\left(\tau + \int_{\tilde{\theta}_0}^{\tilde{\theta}} \zeta_E(\theta) d\theta\right)} = \tilde{r}_0 e^{\rho_E(\theta_j)\left(\tau + \int_{\tilde{\theta}_0}^{\tilde{\theta}} \zeta_E(\theta) d\theta + o(1)\right)}. \quad (10.13)$$

Clearly we obtain from (10.10) and (10.13) the asymptotics

$$(\tilde{\theta} - \theta_j)\tilde{r}^{-\beta} \rightarrow (\tilde{\theta}_0 - \theta_j)\tilde{r}_0^{-\beta} e^{-\lambda \int_{\tilde{\theta}_0}^{\tilde{\theta}} \zeta_E(\theta) d\theta} \text{ for } \tau \rightarrow \infty. \quad (10.14)$$

Motivated by these considerations we introduce a smooth observable $w = w(t, r, \theta)$ on \mathcal{D}_ϵ as follows:

Step I. Define for given $(t, r, \theta) \in \mathcal{D}_\epsilon$

$$E = -\partial_t S(t, r, \theta) (\in I). \quad (10.15)$$

Step II. Introduce

$$\theta_j = \theta_j(E), \quad \beta = \beta(E), \quad \lambda = \lambda(E) = \rho_E(\theta_j)\beta. \quad (10.16)$$

Step III. Put

$$w = (\theta - \theta_j)r^{-\beta} e^{-\lambda \int_{\theta_j}^{\theta} \zeta_E(\theta') d\theta'}. \quad (10.17)$$

From the very construction we see that

$$w = w(t, r, \theta) = t^{-\beta} w(1, \frac{r}{t}, \theta), \quad (10.18)$$

and that w specifies asymptotics of the solution to (10.8) that at time t passes through (r, θ) .

We can now define a comparison dynamics

$$U(t) = U_j(t) : L^2(I \times \mathbb{R}) \rightarrow L^2(\mathbb{R}_x^2) = L^2(\mathbb{R}_+ \times \mathbb{T}; r dr d\theta)$$

as follows:

$$\begin{aligned} [U(t)\phi](r, \theta) &= e^{iS(t,r,\theta)} r^{-1/2} J_t^{1/2}(r, \theta) 1_{\mathcal{D}_{\epsilon,t}}(r, \theta) \\ &\quad \phi(E(t, r, \theta), w(t, r, \theta)), \end{aligned} \quad (10.19)$$

where $E(t, r, \theta) = -\partial_t S(t, r, \theta)$, and $J_t = t^{-\beta-1} J(\frac{r}{t}, \theta)$ is the Jacobian determinant arising from the change of variables which makes $U(t)$ asymptotically isometric. Notice that we may use (10.6) and (10.17) to show that indeed the map $\mathcal{D}_{\epsilon,1} \ni (\frac{r}{t}, \theta) \rightarrow (E, w)(1, \frac{r}{t}, \theta)$ is a diffeomorphism onto its range. Hence by also using (10.18) we have $\lim_{t \rightarrow \infty} \|U(t)\phi\| = \|\phi\|$ for all $\phi \in L^2(I \times \mathbb{R})$.

Formally the “generator” of $U(t)$ is

$$H - 2^{-1}\gamma^2; \quad \gamma = p - \nabla_x S. \quad (10.20)$$

If $\phi \in C_0^\infty(I \times \mathbb{R})$ the right hand side of (10.19) is smooth for t large enough, and we may use the Cook method as in [HS1] to obtain the following wave operator. Let $\mathcal{H}_I = 1_I(H)L^2(\mathbb{R}_x^2)$, and let $M(E)$ denote multiplication by E on $L^2(I \times \mathbb{R})$.

Theorem 10.1. *Suppose there are no resonances (of any order) in I . Suppose (10.1). Then there exists*

$$\Omega_j = s - \lim_{t \rightarrow \infty} e^{itH} U_j(t) : L^2(I \times \mathbb{R}) \mapsto L^2(\mathbb{R}_x^2), \quad (10.21)$$

with $\text{Ran}(\Omega_j) \subseteq \mathcal{H}_I$. Moreover $H\Omega_j = \Omega_j M(E)$.

Remark 10.2. The smallness assumption on I and the choice of $E \in I$ for defining the domain of S above and hence the comparison dynamics $U_j(t)$ are not important. In fact our construction is asymptotically canonical. This means that if we cover a given “large” I free from resonances by “smaller” pieces say $I = \cup I_k$ and choose arbitrarily energies $E_k^0 \in I_k$, then the corresponding dynamics $U_{j,k}(t)$ glue in the following sense: Decompose any given $\phi \in L^2(I \times \mathbb{R})$ as $\phi = \sum \phi_k$ where $\phi_k \in L^2(I_k \times \mathbb{R})$. Then asymptotically as $t \rightarrow \infty$ the state $U_j(t)\phi := \sum U_{j,k}(t)\phi_k$ is independent of the covering and cutting procedures. In particular the definition $\Omega_j \phi = \lim_{t \rightarrow \infty} e^{itH} U_j(t)\phi = \sum \lim_{t \rightarrow \infty} e^{itH} U_{j,k}(t)\phi_k = \sum \Omega_{j,k} \phi_k$ is canonical.

This procedure may be pushed further. Suppose a given open set $\mathcal{C} \subseteq \mathbb{R}$ has a countable covering $\mathcal{C} = \cup I_k$ where each interval meets the requirements of Theorem 10.1. Then we decompose any $\phi \in L^2(\mathcal{C} \times \mathbb{R})$ as above and define $\Omega_j \phi = \sum \Omega_{j,k} \phi_k$.

An alternative procedure of defining a “global” wave operator would be the one of [HS1]. This amounts to introducing the wave operator in terms of an S obtained by gluing the various local pieces considered above together, cf. [HS1, Appendix]; the domain of this S is complicated.

10.2. Wave operator at small energies. We need the condition

$$b(\theta_j) = V'(\theta_j) = 0. \quad (10.22)$$

Notice that this means that θ_j does not depend on $E \in I$. Clearly the case $b = 0$ covered by [HS1, Theorem 1.1] is a particular example. The case $V = 0$ is an example that has not been treated before.

With (10.22) the coordinates x_1, x_2 of (9.1) are independent of E and similarly for the dual coordinates ξ_1, ξ_2 given by writing $\xi = \xi_1 \omega_1 + \xi_2 \omega_2$. We substitute in the expression for the symbol h , Taylor expand up to second order in x_2/x_1 and

ξ_2 , and replace x_1 by $t\xi_1$. The result is the expression

$$\begin{aligned} h(t) &= 2^{-1}\xi_1^2 + h_2(t) + V(\theta_j); \\ h_2(t) &= h_{2,t,\xi_1}\left(\frac{x_1}{t}, \xi_2\right) = 2^{-1}\xi_2^2 + \alpha_1\frac{x_2}{t}\xi_2 + \alpha_2\left(\frac{x_2}{t}\right)^2; \\ \alpha_1 &= -\frac{b'(\theta_j)}{\xi_1}, \\ 2\alpha_2 &= \frac{b'(\theta_j)}{\xi_1}\left(2 + \frac{b'(\theta_j)}{\xi_1}\right) + \frac{V''(\theta_j)}{\xi_1^2}. \end{aligned} \quad (10.23)$$

We denote for fixed $\xi_1 > 0$ the Weyl-quantization of $h_{t,2,\xi_1}$ by $H_2(t)$, and by $U_2(t)$ the corresponding dynamics

$$i\partial_t U_2(t) = H_2(t)U_2(t); \quad U_2(1) = I.$$

It may be written explicitly in terms of

$$H_{2,\xi_1} := 2^{-1}p_2^2 + \alpha_1\frac{x_2p_2 + p_2x_2}{2} + \left(\frac{\alpha_1}{2} - \frac{1}{8} + \alpha_2\right)x_2^2,$$

as

$$U_2(t) = U_{2,\xi_1}(t) = S_{t^{-\frac{1}{2}}} e^{\frac{ix_2^2}{4}} e^{-i(\ln t)H_{2,\xi_1}} e^{-\frac{ix_2^2}{4}},$$

where $S_{t^{-\frac{1}{2}}}$ is the scale transformation

$$S_{t^{-\frac{1}{2}}}g(x_1, x_2) = t^{-\frac{1}{4}}g\left(x_1, \frac{x_2}{\sqrt{t}}\right).$$

The quantum dynamics corresponding to the expression (10.23) is given on the space $L^2(\mathbb{R}_x^2)$ as

$$\bar{U}(t) = e^{-i(t-1)V(\theta_j)} e^{-\frac{itp_1^2}{2}} U_{2,p_1}(t) e^{\frac{ip_1^2}{2}}. \quad (10.24)$$

To get a wave operator with this comparison dynamics we need a small energy assumption. This may be given in terms of the β of (9.2); recall that the quantity $\rho^{-2}\kappa^+$ depends on E . We shall need the condition

$$\Re\beta < -\frac{1}{3} \text{ for } E \in I. \quad (10.25)$$

Clearly the inequality (10.25) is fulfilled for $4\kappa^+ > \rho^2$ in which case β and $\tilde{\beta}$ are complex with real part equal to -2^{-1} .

In Subsection 11.2 we shall need the condition that

$$\beta \neq \tilde{\beta} \text{ for } E \in I \setminus \mathcal{D}, \text{ where } \mathcal{D} \subset I \text{ is discrete,} \quad (10.26)$$

cf. Condition 9.1. For convenience (although not needed) we shall also impose (10.26) in the theorem stated below.

Let $J = \left\{ \xi_1 : \xi_1 > 0, \frac{\xi_1^2}{2} \in I \right\}$, and let \mathcal{H}_I be given as in Theorem 10.1.

Theorem 10.3. *Suppose (10.25) and (10.26). Then there exists*

$$\bar{\Omega}_j = s - \lim_{t \rightarrow \infty} e^{itH} \bar{U}(t) : 1_J(p_1)L^2(\mathbb{R}_x^2) \rightarrow L^2(\mathbb{R}_x^2), \quad (10.27)$$

with $\text{Ran}(\bar{\Omega}_j) \subseteq \mathcal{H}_I$. Moreover $H\bar{\Omega}_j = \bar{\Omega}_j 2^{-1}p_1^2$.

One may prove Theorem 10.3 along the lines of the proof for the potential case of [HS1]. For reference in Subsection 11.2 let us notice the following ingredient of the proof: Let

$$\gamma = p_2 - (\beta + \alpha_1)\frac{x_2}{t} \text{ and } \tilde{\gamma} = p_2 - (\tilde{\beta} + \alpha_1)\frac{x_2}{t}, \quad (10.28)$$

where β , $\tilde{\beta}$ and α_1 are evaluated at $2^{-1}p_1^2$ and p_1 , respectively. Then

$$\gamma(t) := \bar{U}(t)^* \gamma \bar{U}(t) = t^{-\beta} \gamma \text{ and } \tilde{\gamma}(t) := \bar{U}(t)^* \tilde{\gamma} \bar{U}(t) = t^{-\tilde{\beta}} \tilde{\gamma}. \quad (10.29)$$

Due to (10.26) the equations (10.29) can trivially be solved for the quantities $p_2(t)$ and $t^{-1}x_2(t)$ for energies $2^{-1}p_1^2 \notin \mathcal{D}$.

Remarks 10.4. In Theorem 10.3 the interval I is not needed to be “small” (except for the requirement (10.25)). With (10.25) the condition of no resonances of Theorem 10.1 is fulfilled. Under the conditions (10.1), (10.22), (10.25) and (10.26) one can compute explicitly

$$s - \lim_{t \rightarrow \infty} U(t)^* \bar{U}(t) : 1_J(p_1) L^2(\mathbb{R}_x^2) \rightarrow L^2(I \times \mathbb{R}),$$

linking the wave operators of Theorems 10.1 and 10.3.

11. ASYMPTOTIC COMPLETENESS

In this section we discuss asymptotic completeness of the wave operators introduced in Section 10. Now we impose Conditions 2.1 and 2.4, and (6.14).

11.1. Completeness at high energies. With the above assumption we have the following result.

Theorem 11.1. *Under the conditions of Theorem 10.1*

$$\text{Ran}(\Omega_j) = P_j \mathcal{H}_I. \quad (11.1)$$

We shall outline a proof of this theorem using weak propagation estimates similar in spirit to [CHS] and [HS2]. (The proof in the potential case given in [HS1] uses strong propagation estimates.)

Clearly $\text{Ran}(\Omega_j) \subseteq P_j \mathcal{H}_I$. We split the proof of the opposite inclusion into four steps:

I) Preliminary preparation of a state $\psi(t) = e^{-itH} f(H)\psi$; $f \in C_0^\infty(I)$ and $\psi \in P_j \mathcal{H}_I$.

II) Strong localization for the observable γ^2 .

III) Propagation estimates for $U_j(t)\phi$.

IV) Verification of the Cauchy condition.

Step I. We may summarize the estimates of Lemmas 7.2 and 8.3 as follows: Let $L_1(t)$ be the operator with symbol $l_1(t)$ given in terms of small $\epsilon_1, \dots, \epsilon_4 > 0$ as

$$\begin{aligned} l_1(t) &= \tilde{f}(h) F_1 \cdots F_4 \\ &= \tilde{f}(h) F \left(\left| \frac{r}{t} - \rho_j(h) \right| < \epsilon_1 \right) F(|\rho - \rho_j(h)| < \epsilon_2) \\ &\quad F(|\theta - \theta_j(h)| < \epsilon_3) F((b\rho + V')^2 + \eta^2 < \epsilon_4). \end{aligned}$$

Then

$$\psi(t) \approx L_1(t)\psi(t), \quad (11.2)$$

meaning that $\|\psi(t) - L_1(t)\psi(t)\| \rightarrow 0$ for $t \rightarrow \infty$.

It will be important to control the Heisenberg derivative of L_1 . For that purpose we need to replace the last factor F_4 by an equivalent localization described as follows. Let $y = (b\rho + V', \eta)^{\text{tr}}$. Instead of taking $F_4 = F_4(y^2)$ as above we choose $F_4 = F_4(|T_1^{-1}y|^2)$, where T_1 is the matrix

$$T_1 = \begin{pmatrix} \kappa^+ & \kappa^+ \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

given in terms of $\kappa^+ = \kappa^+(h)$ and the two eigenvalues λ_1 and λ_2 in (2.11) (arbitrarily labelled and with $E = h$). This matrix diagonalizes

$$A_1 := \begin{pmatrix} 0 & \kappa^+ \\ -1 & -\rho \end{pmatrix},$$

which arises in the equation

$$\frac{d}{d\tau}y = (A_1 + o(1))y$$

valid for any orbit approaching $z_j(h)$ (in particular having energy $E = h$). We used (2.19). See (2.14) for a similar construction.

Since the eigenvalues of this A_1 coincides with those of (2.11), we see that classical

$$\frac{d}{d\tau}|T_1^{-1}y|^2 \leq 2(\rho\beta + o(1))|T_1^{-1}y|^2; \quad (11.3)$$

in particular $|T_1^{-1}y|^2$ is decreasing.

Clearly there is an analogue of (11.3) for the Poisson bracket. We have the estimate

$$\{h, F_4(|T_1^{-1}y|^2 < \epsilon_4)\} \geq -\epsilon F_4'(\cdot)\rho/r \geq 0 \quad (11.4)$$

for a small $\epsilon > 0$.

Next, we fix the order of scale

$$\epsilon_4 \ll \epsilon \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1, \quad (11.5)$$

for the “new” $L_1(t)$ with symbol

$$\begin{aligned} l_1(t) &= \tilde{f}(h)F_1 \cdots F_4 \\ &= \tilde{f}(h)F\left(\left|\frac{r}{t} - \rho_j(h)\right| < \epsilon_1\right)F(|\rho - \rho_j(h)| < \epsilon_2) \\ &F(|\theta - \theta_j(h)| < \epsilon_3)F(|T_1^{-1}y|^2 < \epsilon_4). \end{aligned} \quad (11.6)$$

To see how (11.5) comes in we look at the contribution to the classical Heisenberg derivative from the factor F_1 . From the proof of (8.7) we learn that

$$\begin{aligned} &\tilde{f}(h)\left(\mathbf{d}F_1\left(\left|\frac{r}{t} - \rho_j(h)\right| < \epsilon_1\right)\right)F_2 \cdots F_4 \\ &\geq -\epsilon t^{-1}\tilde{f}(h)F_1'(\cdot)F_2 \cdots F_4 \geq 0 \end{aligned} \quad (11.7)$$

for $\epsilon_2 < \epsilon_1/2$ and for $\epsilon > 0$ sufficiently small.

We also learn from the proof of (8.7) that the contribution from the Poisson bracket from the factor F_2 is negligible

$$\tilde{f}(h)F_1(\mathbf{d}F_2(|\rho - \rho_j(h)| < \epsilon_2))F_3F_4 = \mathcal{O}(t^{-2}) \quad (11.8)$$

for $\epsilon_3, \epsilon_4 \ll \epsilon_2$.

Finally, the contribution from the Poisson bracket from the factor F_3 is negligible

$$\tilde{f}(h)F_1F_2(\mathbf{d}F_3(|\theta - \theta_j(h)| < \epsilon_3))F_4 = \mathcal{O}(t^{-2}) \quad (11.9)$$

for $\epsilon_4 \ll \epsilon_3$, see (8.3).

We conclude that

$$\mathbf{D}L_1(t) \geq \mathcal{O}(t^{-2}), \quad (11.10)$$

and the estimates

$$\int_1^\infty t^{-1}|\langle \text{Op}^w(b_1) \rangle_t| dt \leq C\|\psi\|^2; \quad b_1 = -\tilde{f}(h)F_1'(\cdot)F_2 \cdots F_4, \quad (11.11)$$

and

$$\int_1^\infty t^{-1}|\langle \text{Op}^w(b_2) \rangle_t| dt \leq C\|\psi\|^2; \quad b_2 = -\tilde{f}(h)F_1 \cdots F_3F_4'(\cdot). \quad (11.12)$$

We used (11.7) and (11.4).

Step II. Classically $\gamma = \xi - \nabla_x S$ vanishes along any orbit $x(t)$ associated with the channel we study. Here $S = S(t, r, \theta)$ is defined in a domain \mathcal{D}_ϵ as described in Subsection 10.1. We compute

$$\frac{d}{dt}\gamma = -(\partial_x \partial_\xi h + \nabla_x^2 S + o(t^{-1}))\gamma.$$

Notice that this follows from a classical analogue of (11.2) since (recall that \mathbf{d} is the classical Heisenberg derivative)

$$l_1(t)\mathbf{d}\gamma = -l_1(t) (\partial_x \partial_\xi h + \nabla_x^2 S + t^{-1} \mathcal{O}(\epsilon_1)) \gamma. \quad (11.13)$$

Here $|\mathcal{O}(\epsilon_1)| \leq C\epsilon_1$ uniformly in t .

We are motivated to calculate the matrix

$$A_2 = -t (\partial_x \partial_\xi h + \nabla_x^2 S_E) \quad (11.14)$$

at the particular energy

$$I \ni E = h = -\partial_t S(t, r, \theta).$$

Represented in the coordinates x_1, x_2 of (9.1) with $E = h$ we may compute using (10.4) and (10.7)

$$A_2 = - \begin{pmatrix} (1 + \sigma^2)^{-1} & \sigma(1 + \sigma^2)^{-1} + b\rho^{-1} \\ \sigma(1 + \sigma^2)^{-1} & -\tilde{\beta} + \sigma^2(1 + \sigma^2)^{-1} \end{pmatrix}; \quad \sigma = \rho\beta b(\kappa^+)^{-1}.$$

Here the various quantities are computed at $(E, \theta) = (h, \theta_j(h))$.

The eigenvalues of A_2 are -1 and $\tilde{\beta}$. Let T_2 denote a real 2×2 -matrix that diagonalizes A_2 . We obtain

$$l_1(t)\mathbf{d}|T_2^{-1}\gamma|^2 \leq 2t^{-1}(\tilde{\beta} + \mathcal{O}(\epsilon_1))l_1(t)|T_2^{-1}\gamma|^2. \quad (11.15)$$

Clearly (11.15) yields the decay

$$\gamma(t) = \mathcal{O}(t^{\tilde{\beta} + \epsilon'}); \quad \epsilon' > 0, \quad (11.16)$$

along the orbit $x(t)$.

We aim at a similar statement in quantum mechanics. (Notice that by (11.16) γ^2 is integrable.) For that we pick $\tilde{f} \in C_0^\infty(I)$ such that $0 \leq \tilde{f} \leq 1$ and $\tilde{f} = 1$ in a neighborhood of $\text{supp } \tilde{f}$. Let $l_2(t)$ be given as in (11.6) with \tilde{f} and ϵ_k replaced by \tilde{f} and $2\epsilon_k$; $k = 1, \dots, 4$, respectively. We consider localization in terms of the operator $P = P(t) = \text{Op}^w(p(t))$ where

$$p(t) = l_2(t)(T_2^{-1}\gamma)^2 l_2(t). \quad (11.17)$$

Notice that, for example $(1 - l_2(t))l_1(t) = 0$.

Using (11.2) and the above calculations for the classical case we may mimic the proof of [HS2, Proposition 5.1] (see also the one of [CHS, Proposition 7.1]). We obtain the following strong localization statement. Fix $\epsilon' > 0$ such that $\epsilon' < -2\tilde{\beta}(E) - 1$ for all $E \in \text{supp } \tilde{f}$. Then for ϵ_1 small enough

$$\psi(t) \approx F(t^{1+\epsilon'} P(t) < 1) L_1(t) \psi(t). \quad (11.18)$$

The proof yields the integral estimate

$$\int_1^\infty t^{-1} \left\langle L_1(-F')(t^{1+\epsilon'} P < 1) L_1 \right\rangle_t dt \leq C \|\psi\|^2, \quad (11.19)$$

in agreement with (11.15).

Step III. Recall that $U_j(t)$ is defined in terms of (a small) $\epsilon > 0$, I and the corresponding domain $\mathcal{D}_{\epsilon,t}$. We let $\epsilon_1, \dots, \epsilon_4$ and $\epsilon' > 0$ be given in agreement with Step II.

The integral estimates (11.11), (11.12) and (11.19) remain true upon replacing ψ and $\psi(t) = e^{-itH} f(H)\psi$ by $\phi \in C_0^\infty(I \times \mathbb{R})$ and $U_j(t)\phi$, respectively.

To see this we introduce a smooth function $g \in C_0^\infty(\mathcal{D}_{\epsilon,1})$ such that with $g_t(r, \theta) = g(\frac{r}{t}, \theta)$ the following support property holds for all multi-indices $|\alpha| \geq 0$:

$$l_1(t) \partial_{t,x}^\alpha (1 - g_t) = 0. \quad (11.20)$$

We consider the expectation of the family of uniformly bounded operators

$$\Phi(t) = g_t L_1(t) F(P(t)t^{1+\epsilon'} < 1) L_1(t) g_t \quad (11.21)$$

in the states $U_j(t)\phi$. We compute using (11.20) and the expression (10.20) for the “generator” (notice that $g_t \in C_0^\infty(\mathcal{D}_{\epsilon,t})$)

$$L_1(t) i \partial_t g_t U_j(t)\phi = L_1(t) (H - 2^{-1}P(t)) g_t U_j(t)\phi + O(t^{-\infty}),$$

and from the derivation of the bounds for ψ we see that indeed

$$\frac{d}{dt} \Phi(t) + i[H - 2^{-1}P(t), \Phi(t)] \geq \mathcal{O}(t^{-1-\epsilon'}),$$

expressing the “attractiveness” of all localizations. Since the right hand side is integrable we obtain the integral estimates for $g_t U_j(t)\phi$ and hence for $U_j(t)\phi$.

Step IV. We need to verify the existence of the limit

$$\lim_{t \rightarrow \infty} U_j(t)^* e^{-itH} \psi; \quad \psi(t) = e^{-itH} f(H)\psi.$$

As in Step III we consider $\phi \in C_0^\infty(I \times \mathbb{R})$ and its approximate evolution $U_j(t)\phi$. We may compute the derivative

$$\frac{d}{dt} \langle U_j(t)\phi, \Phi(t)\psi(t) \rangle,$$

where $\Phi(t)$ is given by (11.21), as before. Using the integral estimates (11.11), (11.12) and (11.19) for $\psi(t)$ and the similar bounds for $U_j(t)\phi$, we obtain the estimate

$$\int_T^\infty \left| \frac{d}{dt} \langle U_j(t)\phi, \Phi(t)\psi(t) \rangle \right| dt \leq o(T^0) \|\phi\|,$$

as $T \rightarrow \infty$.

The proof is complete.

11.2. Completeness at low energies. With the same assumptions as those of Subsection 10.2 we have the following result.

Theorem 11.2. *Under the conditions of Theorem 10.3*

$$\text{Ran}(\bar{\Omega}_j) = P_j \mathcal{H}_I. \quad (11.22)$$

We shall outline a proof of this result using weak propagation estimates. (The one for the potential case in [HS1] uses strong propagation estimates.) Since we may follow the scheme of the proof of Theorem 11.1 we shall be brief.

Due to the discreteness assumption of the \mathcal{D} in (10.26) we may assume that $I \cap \mathcal{D} = \emptyset$.

Clearly $\text{Ran}(\bar{\Omega}_j) \subseteq P_j \mathcal{H}_I$. For the opposite inclusion, we follow Step I literally; the function $l_1(t)$ given by (11.6) will be used below. Step II is different. (Notice that the γ of (10.28) differs from the γ of the proof of Theorem 11.1.) We introduce the symbol $l_2(t)$ as before, but instead of (11.17) we define

$$p(t) = l_2(t) \left(\gamma^2 + \tilde{\gamma}^2 + (\xi_1 - t^{-1}x_1)^2 \right) l_2(t), \quad (11.23)$$

where γ and $\tilde{\gamma}$ are given by (10.28).

Fix $\epsilon' \in (0, \frac{1}{2})$ such that

$$3\Re\beta(E) < -1 - \epsilon' \text{ for } E \in \text{supp } \tilde{f}. \quad (11.24)$$

We notice that

$$t^{\frac{2}{3}(1+\epsilon')} p(t) \in S_{\text{unif}}(1, t^{\frac{2}{3}\epsilon' - \frac{4}{3}} dx^2 + t^{\frac{2}{3}(1+\epsilon')} d\xi^2), \quad (11.25)$$

using here a uniform symbol class; see [HS2, Definition 3.1] and the discussion after the definition.

The “Planck’s constant” $t^{-\frac{1}{3} + \frac{2}{3}\epsilon'} \rightarrow 0$.

We may mimic the proof of [HS2, Lemma 4.4] to obtain the following analogue of (11.18):

$$\psi(t) \approx \text{Op}^w(F(t^{\frac{2}{3}(1+\epsilon')}p(t) < 1))L_1(t)\psi(t). \quad (11.26)$$

The analogue of (11.19) reads:

$$\int_1^\infty t^{-1} |\langle L_1 \text{Op}^w(F'(t^{\frac{2}{3}(1+\epsilon')}p(t) < 1))L_1 \rangle_t| dt \leq C \|\psi\|^2. \quad (11.27)$$

For completeness of presentation we remark that one may also derive statements like (11.26) and (11.27) using the functional calculus as done for (11.18) and (11.19). We find the pseudodifferential approach somewhat simpler; the reader might not. In any case a pseudodifferential approach does not work for (11.18) and (11.19), cf. a discussion in [HS2].

Now returning to the remaining steps, we need in Step III to show similar integral estimates as those from Step I and (11.27) for $\bar{U}(t)\phi$ where $\phi \in 1_J(p_1)L^2(\mathbb{R}_x^2)$. In fact it suffices to have these estimates for $\phi \in g(p_1)\mathcal{S}(\mathbb{R}_x^2)$, where $g \in C_0^\infty(J)$ and $\mathcal{S}(\cdot)$ denotes the Schwartz space. For such states we have $i\partial_t \bar{U}(t)\phi = \bar{H}(t)\bar{U}(t)\phi$ and we may use that the difference between the generators, $\bar{H}(t) - H$, is integrable in combination with the localization (11.26).

For Step IV we proceed as before.

The outline of proof is complete.

12. THE GENERAL CLASSICAL CASE

We shall look at a periodic b with $\int_0^{2\pi} b \, d\theta \leq 0$. (Possibly by replacing $b(\theta) \rightarrow -b(\frac{\pi}{2} - \theta)$ this assumption is for free.)

We impose the following condition for a given open interval I of energies.

Condition 12.1 (Torus).

$$\inf I \geq \max V. \quad (12.1)$$

With this assumption the reduced phase space $\mathbb{T}_E^* := \{z \in \mathbb{T}^* \mid h(z) = E\}$ is a torus. The case of energies $E < \max V$ is somewhat easier to treat, see Remark 12.5.

We have the following result on the classical dynamics for $\tau \rightarrow +\infty$. A similar result holds for $\tau \rightarrow -\infty$.

Let for $E \in I$

$$\mathbb{T}^{*,+}(E) = \left\{ z \in \mathbb{T}_E^* \mid \rho > -\sqrt{2(E - V(\theta))} \right\}, \quad (12.2)$$

and

$$\mathbb{T}^{*,-}(E) = \left\{ z \in \mathbb{T}_E^* \mid \rho < \sqrt{2(E - V(\theta))} \right\}. \quad (12.3)$$

Clearly the topological structure of both of the sets $\mathbb{T}^{*,+}(E)$ and $\mathbb{T}^{*,-}(E)$ is an annulus.

The following result involves the notion of *regular solutions* $\rho = \rho_r(\theta)$ of the equation $\rho' = b + \eta$ which may exist for $\int_0^{2\pi} b \, d\theta < 0$. For later purposes let us also introduce the set

$$\mathcal{E} = \left\{ E > \max V \mid \text{a } 2\pi\text{-periodic solution } \rho_E \text{ exists,} \right. \\ \left. 2(E - V) > \rho_E^2, \int_0^{2\pi} \frac{\rho_E}{\sqrt{2(E - V) - \rho_E^2}} \, d\theta > 0 \right\}. \quad (12.4)$$

Here by definition the function ρ_E solves

$$\rho_E' = b + \eta_E; \quad \eta_E = \sqrt{2(E - V) - \rho_E^2}; \quad (12.5)$$

it is called an *upper regular* solution. If we drop the sign condition of the integral of (12.4) and only require that $2(E - V) > \rho_E^2$ for a solution of (12.5), then we say that it is a *regular* solution.

The result stated below also involves the notion of *singular solutions/cycles* $\rho = \rho_r(\theta)$ of the equation $\rho' = b + \eta$. Let us introduce the set $\mathcal{F}_1(E) = \{\theta \in \mathbb{T} \mid \exists z \in \mathcal{F}(E) : z = (\theta, 0, \rho)\}$.

We shall discuss two types of singular cycles which we call upper and lower singular cycles, respectively.

A 2π -periodic C^1 -solution ρ_E to (12.5) is called an *upper singular cycle* if $S^+ := \{\theta \in \mathcal{F}_1(E) \mid \eta_E(\theta) = 0\} \neq \emptyset$ and this set $S^+ = \{\theta \in \mathcal{F}_1(E) \mid \rho_E(\theta) = \sqrt{2(E - V(\theta))}\}$. With the latter assumption and (2.9), $\kappa^+(\theta_0) < 0$ for all $\theta_0 \in S^+$ (thus saddles).

A 2π -periodic C^1 -solution ρ_E of (12.5) is called a *lower singular cycle* if $S^- := \{\theta \in \mathcal{F}_1(E) \mid \eta_E(\theta) = 0\} \neq \emptyset$ and $S^- = \{\theta \in \mathcal{F}_1(E) \mid \rho_E(\theta) = -\sqrt{2(E - V(\theta))}\}$. Clearly with (2.10), $\kappa^-(\theta_0) < 0$ for all $\theta_0 \in S^-$ (thus saddles again).

If S^+ (or S^-) only consists of one point for a given singular cycle, then the cycle corresponds to a homoclinic orbit for the dynamics (2.6).

Proposition 12.2. *Suppose Condition 12.1, (2.9) and (2.10) at some $E \in I$, and that $\gamma = (\theta, \eta, \rho)$ is an arbitrary classical orbit with energy E . Then one of the following cases occurs:*

- i) *There exist sequences $\tau_n^- \rightarrow +\infty$ and $\tau_n^+ \rightarrow +\infty$ such that*

$$\rho(\tau_n^-) = -\sqrt{2(E - V(\theta(\tau_n^-)))}, \quad (12.6)$$

and

$$\rho(\tau_n^+) = +\sqrt{2(E - V(\theta(\tau_n^+)))}, \quad (12.7)$$

- ii) *The set $\mathcal{F}(E) \neq \emptyset$, and there exists $z \in \mathcal{F}(E)$ such that $\gamma(\tau) \rightarrow z$ for $\tau \rightarrow +\infty$.*
 iii) *There exists a regular solution $\rho = \rho_r(\theta)$ to the equation $\rho' = b + \eta$, such that*

$$\lim_{\tau \rightarrow +\infty} |\rho(\tau) - \rho_r(\theta(\tau))| = 0; \quad \lim_{\tau \rightarrow +\infty} |\eta(\tau) - \eta_r(\theta(\tau))| = 0. \quad (12.8)$$

- iv) *There exists a singular cycle ρ_s such that $(\eta_s = \sqrt{2(E - V) - \rho_s^2})$*

$$\lim_{\tau \rightarrow +\infty} |\rho(\tau) - \rho_s(\theta(\tau))| = 0; \quad \lim_{\tau \rightarrow +\infty} |\eta(\tau) - \eta_s(\theta(\tau))| = 0. \quad (12.9)$$

Proof. Using that $\frac{d}{d\tau}(\rho - \tilde{b}) = \eta^2$ (cf. the proof of Proposition 2.3) we have two possibilities, either 1) $\lim_{\tau \rightarrow +\infty} \theta(\tau) = +\infty$, or 2) $\theta(\tau)$ stays bounded near $+\infty$. (Notice that 1) occurs precisely if $\int_0^\infty \eta^2 d\tau = \infty$. To show these assertions it is convenient to distinguish between the case where the flux is zero and the case where it is nonzero.) In the case of 2) we may show that ii) occurs by mimicking the proof of Proposition 2.3. So we can assume 1).

Suppose also that (12.6) is false for all sequences $\tau_n^- \rightarrow +\infty$. Then $\gamma(\tau)$ takes values in the set $\mathbb{T}^{*,+}(E)$ given by (12.2) for all large τ 's. The topological structure of $\mathbb{T}^{*,+}(E)$ is an annulus and therefore we deduce (using 1)) that $\eta > 0$ eventually. Next we may write $\tau = \tau(\theta)$ using the equation $\theta' = \eta$, and thus $\rho = \rho(\theta)$. Let

$$\rho_p(\theta) = \lim_{n \rightarrow \infty} \rho(\theta + 2\pi n); \quad (12.10)$$

notice that the sequence is monotone. Clearly $\rho = \rho_p$ is a periodic solution to $\rho' = b + \eta_p$; $\eta_p = \sqrt{2(E - V) - \rho^2}$. If η_p does not have zeroes, iii) occurs. If η_p has a zero, then iv) occurs.

If (12.7) is false for all sequences $\tau_n^+ \rightarrow +\infty$ we look at the ‘‘annulus’’ $\mathbb{T}^{*,-}(E)$ given by (12.3) and argue similarly. \square

Remark 12.3. There is a version of Proposition 12.2 for negative times; it is given by replacing $+\infty$ by $-\infty$ at all occurrences. The proof is similar.

Remark 12.4. In the case of zero flux, Proposition 12.2 iii) and iv) do not occur (seen by integrating (12.5)). Moreover it follows from Proposition 2.3 that i) does not occur in this case. Whence ii) and only ii) occurs in this case.

Remark 12.5. Suppose the non-critical condition (2.4), (2.9) and (2.10) for an energy $E < \max V$. Then for all classical orbits with this energy Proposition 12.2 ii) occurs. This follows readily from the proof of Proposition 12.2.

12.1. The set \mathcal{E} . Motivated by Proposition 12.2 iii) we shall in this subsection study the set \mathcal{E} . The analysis does not use (2.9) nor (2.10).

Let us show that the periodic orbit ρ_E in the definition of \mathcal{E} is unique: Suppose ρ_1 and ρ_2 are two solutions obeying the above conditions for the same E . Let θ_2 be a solution to $\theta'_2 = \eta_2(\theta_2)$. Let $r_2(\theta) = \exp(\int \rho_2/\eta_2 d\theta)$. Look at $A_2(\theta_2) = r_2(\theta_2)(\rho_1 - \rho_2)(\theta_2)$. We compute and estimate $A'_2 = r_2(\rho_2(\rho_1 - \rho_2) + \eta_2(\eta_1 - \eta_2)) \leq 0$. Thus A_2 is bounded from above. Since $r_2(\theta_2) \rightarrow +\infty$ and the ρ 's are periodic we conclude that $\rho_1 \leq \rho_2$. By symmetry we see that $\rho_1 = \rho_2$.

Proposition 12.6. *Suppose $V = 0$ and $\mathcal{E} \neq \emptyset$. We have*

- 1) \mathcal{E} is an open interval $\mathcal{E} = (E_d, E_e)$, $E_d > 0$.
- 2) $\lim_{E \rightarrow E_d} \partial_E \rho_E(\theta) = +\infty$.
- 3) $\lim_{E \rightarrow E_d} \int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta = 0$.
- 4) $\partial_E \rho_E(\theta) > 1/\sqrt{2E}$.
- 5) $\partial_E^2 \rho_E(\theta) < 0$.
- 6) $\partial_E \frac{\rho_E}{\eta_E} > 0$.
- 7) *The function $E \rightarrow \rho_E(\theta) - \sqrt{2E}$ is increasing.*
- 8) $\lim_{E \rightarrow E_e} \int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta = +\infty$.
- 9) $\eta_E(\theta) \leq \max |b|$.
- 10) *Suppose $2\tilde{E} := 2E - \max b^2 > 0$. Then $\rho_E \geq \sqrt{2\tilde{E}}$ and $\partial_E \rho_E(\theta) \leq 1/\sqrt{2\tilde{E}}$. In particular $\lim_{E \rightarrow E_e} \partial_E \rho_E(\theta) = 0$ if $E_e = \infty$.*

Proof. As in the proof of [CHS, Proposition 2.4] we see that \mathcal{E} is open. Take for the moment a maximal subinterval $(E_d, E_e) \subseteq \mathcal{E}$. By integrating (12.5) we see that $\sqrt{2E_d} \geq (2\pi)^{-1} \int_0^{2\pi} -b(\theta) d\theta > 0$; in particular $E_d > 0$.

We have the formula

$$\partial_E \rho(\theta) = \int_{-\infty}^{\theta} \frac{1}{\eta(\theta')} e^{\int_{\theta'}^{\theta} \frac{\rho}{\eta} d\bar{\theta}} d\theta'. \quad (12.11)$$

In particular

$$\partial_E \rho(\theta) > \int_{-\infty}^{\theta} \frac{1}{\eta(\theta')} e^{\int_{\theta'}^{\theta} \frac{\sqrt{2E}}{\eta} d\bar{\theta}} d\theta' = \frac{1}{\sqrt{2E}}, \quad (12.12)$$

showing 4).

Rewriting 4) as $\partial_E(\rho_E(\theta) - \sqrt{2E}) > 0$ yields 7).

As for 6) we compute $\partial_E \frac{\rho_E}{\eta_E} = (2E \partial_E \rho_E - \rho_E)/\eta^3$. Using (12.12) again this leads to $\partial_E \frac{\rho_E}{\eta_E} \geq (\sqrt{2E} - \rho_E)/\eta^3 > 0$.

As for 5) we refer to the proof of [CHS, (2.19)].

By monotonicity, cf. (12.12), there exist

$$\rho_d(\theta) = \lim_{E \rightarrow E_d} \rho_E(\theta), \quad \eta_d(\theta) = \lim_{E \rightarrow E_d} \eta_E(\theta),$$

and if $E_e < \infty$

$$\rho_e(\theta) = \lim_{E \rightarrow E_e} \rho_E(\theta), \quad \eta_e(\theta) = \lim_{E \rightarrow E_e} \eta_E(\theta).$$

Let η_j denote either η_d or η_e (assuming $E_e < \infty$ for the latter). Suppose $\eta_j(\theta_0) = 0$ for some θ_0 . Then by (12.5), $b(\theta_0) = 0$. We claim that

$$\int_0^{2\pi} \frac{1}{\eta_j} d\theta = \infty. \quad (12.13)$$

Suppose not, then we learn from writing $\eta^2 = (\sqrt{2E} - \rho)(\sqrt{2E} + \rho)$ near $\theta = \theta_0$ (with $\eta = \eta_j$ and $E = E_j$) and using (12.5) that

$$\eta^2 \leq C \left((\theta - \theta_0)^2 + \left| \int_{\theta_0}^{\theta} \eta(\theta') d\theta' \right| \right) =: g(\theta). \quad (12.14)$$

Let us look at $g(\theta)$ to the right of θ_0 . We have

$$g' \leq 2C(\theta - \theta_0) + C\sqrt{g}.$$

Let $f(\theta) = K(\theta - \theta_0)^2$ with $K > 0$ taken such that

$$f' = 2C(\theta - \theta_0) + C\sqrt{f}.$$

Then

$$(f - g)' \geq C(\sqrt{f} - \sqrt{g}) = \frac{C}{\sqrt{f} + \sqrt{g}}(f - g).$$

Since $\int \frac{C}{\sqrt{f} + \sqrt{g}} d\theta < \infty$ we conclude that $\sqrt{K}|\theta - \theta_0| \geq \sqrt{f} \geq \sqrt{g} \geq \eta$. This contradicts the finiteness assumption, and therefore (12.13) holds.

Let us show 2) and 3). We first show that for $\rho_d(\theta) \neq \sqrt{2E_d}$ for all θ . Indeed if $\rho_d(\theta) = \sqrt{2E_d}$ then $b(\theta) = 0$, and by 7) for $E' < E$

$$0 > \rho_E(\theta) - \sqrt{2E} \geq \rho_{E'}(\theta) - \sqrt{2E'},$$

yielding a contradiction by letting $E' \rightarrow E_d$.

If $\rho_d(\theta) = -\sqrt{2E_d}$ for some θ , then $\lim_{E \rightarrow E_d} \int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta = -\infty$ by (12.13) and Fatou's Lemma, which conflicts the fact that $\int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta > 0$.

But then

$$\int_0^{2\pi} \frac{\rho_d}{\eta_d} d\theta = 0, \quad (12.15)$$

since the integral exists and E_d is a positive endpoint of the maximal interval. This shows 3), and 2) follows from (12.11), 3) and Fatou's Lemma.

We can now prove that \mathcal{E} is an interval. Let E_d^1 and E_d^2 be the left endpoints of two maximal intervals. Similar notation is used for the corresponding periodic functions ρ and η . Let θ_2 be a solution to $\theta'_2 = \eta_2(\theta_2)$. Let $r_2(\theta) = \exp(\int \rho_2/\eta_2 d\theta)$. Look at $A_2(\theta_2) = r_2(\theta_2)(\rho_1 - \rho_2)(\theta_2)$. We compute $A'_2 = r_2(\rho_2(\rho_1 - \rho_2) + \eta_2(\eta_1 - \eta_2))$. If $E_d^1 < E_d^2$ we see that $A'_2 \leq -\epsilon$ contradicting that A_2 is bounded. Thus (by symmetry) $E_d^1 = E_d^2$. This shows that \mathcal{E} is an interval.

The statement 9) follows readily using (12.5) and the periodicity, and 10) from 9) and by estimating (12.11), cf. (12.12).

Finally, to show 8) we can assume that E_e is finite, cf. 9) and 10). By 6) the limit

$$\lim_{E \rightarrow E_e} \int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta \in (0, \infty]$$

exists. Suppose it is finite. Then by the Lebesgue monotone convergence theorem and (12.13) we see that $\eta_e(\theta) > 0$ for all θ , contradicting the maximality of E_e . \square

Small modifications of the above proof yield the following results in the general case.

Proposition 12.7. *Suppose $\mathcal{E} \neq \emptyset$. We have*

1) \mathcal{E} is an open set with $E_d := \inf \mathcal{E} > \min V$.

- 2) $\partial_E \rho_E(\theta) > 1/\sqrt{2(E - \min V)}$.
- 3) $\partial_E^2 \rho_E(\theta) < 0$.
- 4) The function $E \rightarrow \rho_E(\theta) - \sqrt{2(E - \min V)}$ is increasing.
- 5) Suppose $\hat{I} = (\hat{E}_d, \hat{E}_e)$ is a maximal subinterval of \mathcal{E} . Then

$$\lim_{E \rightarrow (\hat{E}_e)^-} \int_0^{2\pi} \frac{\rho_E}{\eta_E} d\theta = +\infty.$$

- 6) $\eta_E(\theta) \leq \max |b|$.
- 7) Suppose $2\tilde{E} := 2(E - \max V) - \max b^2 > 0$. Then $\rho_E \geq \sqrt{2\tilde{E}}$ and $\partial_E \rho_E(\theta) \leq 1/\sqrt{2\tilde{E}}$.

Under the conditions of Proposition 12.7 it is an open problem whether \mathcal{E} is an interval.

We notice that in the case $b < 0$ and $V = 0$ studied in [CHS] the set $\mathcal{E} = (E_d, \infty)$. As the following example shows there are other cases where $\mathcal{E} \neq \emptyset$.

Example 12.8. Take $V = 0$, $b(\theta) := -c < 0$, if $0 \leq \theta \leq 2\pi - \epsilon$, and $-c \leq b(\theta) \leq c$. Whence we do not have a sign assumption on the interval $(2\pi - \epsilon, 2\pi)$. In particular $\mathcal{F}(E) \neq \emptyset$ is allowed. For this class of examples:

For all $E > c^2/2$ we can find $\epsilon = \epsilon_E > 0$ small enough such that $E \in \mathcal{E}$.

This may be proved by elementary differential inequality techniques.

On the other hand, if $V = 0$ and $b(\theta) \geq 0$ in an interval $I = [\theta_1, \theta_2]$ of length $|I| = \pi$ (no sign assumption outside I), then indeed $\mathcal{E} = \emptyset$: Suppose on the contrary that $E \in \mathcal{E}$ exists. Then the corresponding periodic solution obeys $\rho' \geq \eta = \sqrt{2E - \rho^2}$ on I . Hence

$$\begin{aligned} \pi = |I| &\leq \int_I \frac{\rho'}{\sqrt{2E - \rho^2}} d\theta = \int_{\rho(\theta_1)}^{\rho(\theta_2)} \frac{1}{\sqrt{2E - \rho^2}} d\rho \\ &= \sin^{-1} \left(\frac{\rho(\theta_1)}{\sqrt{2E}} \right) - \sin^{-1} \left(\frac{\rho(\theta_2)}{\sqrt{2E}} \right) < \pi. \end{aligned} \quad (12.16)$$

Furthermore in the more general situation, $V = 0$ and $b(\theta) \geq 0$ in an interval $I = [\theta_1, \theta_2]$; $\theta_1 < \theta_2$, the same estimation as above leads to

$$|I| \leq \sin^{-1} \left(\frac{\rho(\theta_1)}{\sqrt{2E}} \right) - \sin^{-1} \left(\frac{\rho(\theta_2)}{\sqrt{2E}} \right). \quad (12.17)$$

Since $\rho(\theta_j) \geq \sqrt{2E - \max b^2}$, cf. Proposition 12.6 10), the right hand side is close to $\frac{\pi}{2} - \frac{\pi}{2} = 0$ for E large, in particular $< |I|$. So ρ does not exist for E large.

Remark 12.9. For $E > E_d$ in the case where $\mathcal{E} \neq \emptyset$ one easily verify that Proposition 12.2 i) does not occur by checking that $e^{\int \rho(\rho - \rho_{E'})}$ is increasing in τ for $E' < E$ in \mathcal{E} .

12.2. Singular cycles. Motivated by Proposition 12.2 iv) and Proposition 12.6 8) (notice that $\rho_e = \lim_{E \rightarrow E_e} \rho_E$ indeed is an upper singular cycle) we shall in this subsection study singular cycles under slightly more restrictive conditions than in Proposition 12.6.

12.2.1. Uniqueness of the upper singular cycle.

Proposition 12.10. *Suppose $V = 0$, Condition 12.1 and (2.9) for all $E \in I$. Then there is at most one energy in I for which an upper singular cycle exists, and this solution is uniquely determined.*

Proof. Let $\rho_1 = \rho_{E_1}$ and $\rho_2 = \rho_{E_2}$ be two given upper singular cycles. We need to show that $\rho_1 = \rho_2$. Let $\eta_j(\theta) = \sqrt{2E_j - \rho_j^2}$ be the corresponding transversal velocity and $S_j := \{\theta \in \mathcal{F}_1(E_j) \mid \eta_j(\theta) = 0\}$. We consider three cases.

Case I Suppose $E_1 < E_2$ and that

$$S_1 \cap S_2 \neq \emptyset.$$

We introduce $\sigma_j = \rho_j - \sqrt{2E_j}$, and obtain from (12.5) that

$$\sigma'_j = b + f_j(\sigma_j); \quad f_j(\sigma) = \sqrt{-2\sqrt{2E_j}\sigma - \sigma^2}. \quad (12.18)$$

We notice the following formulas near any $\theta_0 \in S_1 \cap S_2$.

$$\begin{aligned} \sigma_j(\theta) = & \left\{ b'(\theta_0) + \frac{\sqrt{2E_j}}{2} \left(-1 + \sqrt{1 - \frac{4b'(\theta_0)}{\sqrt{2E_j}}} \right) \right\} \frac{(\theta - \theta_0)^2}{2} \\ & + O((\theta - \theta_0)^3) \text{ for } \theta \downarrow \theta_0, \end{aligned} \quad (12.19)$$

$$\begin{aligned} \sigma_j(\theta) = & \left\{ b'(\theta_0) - \frac{\sqrt{2E_j}}{2} \left(1 + \sqrt{1 - \frac{4b'(\theta_0)}{\sqrt{2E_j}}} \right) \right\} \frac{(\theta - \theta_0)^2}{2} \\ & + O((\theta - \theta_0)^3) \text{ for } \theta \uparrow \theta_0. \end{aligned} \quad (12.20)$$

We deduce from (12.19) and (12.20) that for some $\epsilon > 0$

$$\sigma_2(\theta) \begin{cases} > \sigma_1(\theta) \text{ provided } 0 < \theta - \theta_0 < \epsilon \\ < \sigma_1(\theta) \text{ provided } 0 > \theta - \theta_0 > -\epsilon \end{cases}. \quad (12.21)$$

Let us also notice that away from joint zeros of the σ_j 's, that is away from the set $S_1 \cap S_2$,

$$(\sigma_2 - \sigma_1)' \geq -h_2(\sigma_2 - \sigma_1); \quad h_2 = h_2(\theta) = \frac{2\sqrt{2E_2} + \sigma_2 + \sigma_1}{f_2(\sigma_1) + f_2(\sigma_2)}. \quad (12.22)$$

We learn from (12.22) that the function

$$\theta \rightarrow e^{\int_{\tilde{\theta}}^{\theta} h_2 d\theta} (\sigma_2 - \sigma_1)(\theta)$$

is non-decreasing on any interval $[\tilde{\theta}, \tilde{\theta}]$ disjoint from $S_1 \cap S_2$, in particular on some set of the form $(\theta_0, \tilde{\theta}_0)$, where $\theta_0, \tilde{\theta}_0 \in S_1 \cap S_2$. Here we use the periodicity of the η_j 's. (If $S_1 \cap S_2 \subset \mathbb{T}$ only consists of one point θ_0 this amounts to looking at one entire revolution, $\tilde{\theta}_0 = \theta_0 + 2\pi$.) By combining this monotonicity with the upper part of (12.21) at the angle θ_0 and the lower part at $\theta_0 \rightarrow \tilde{\theta}_0$ we arrive at a contradiction.

Case II Suppose $E_1 \leq E_2$ and that

$$S_1 \cap S_2 = \emptyset.$$

Introduce again $\sigma_j = \rho_j - \sqrt{2E_j}$. Pick arbitrary $\theta_1 \in S_1$ and $\theta_2 \in S_2$. We may assume that $\theta_2 < \theta_1$ (otherwise replace $\theta_1 \rightarrow \theta_1 + 2\pi$). Clearly $\sigma_2(\theta_2) > \sigma_1(\theta_2)$ and $\sigma_2(\theta_1) < \sigma_1(\theta_1)$. By using (12.22) to the interval $[\theta_2, \theta_1]$ as before we conclude that $\sigma_2(\theta_1) > \sigma_1(\theta_1)$, a contradiction.

Case III Suppose $E_1 = E_2$. We modify the arguments for uniqueness of outgoing spirals given before Proposition 12.6: Let $\theta_2^-, \theta_2^+ \in S_2$ be given such that $\theta_2^- < \theta_2^+$ and $(\theta_2^-, \theta_2^+) \cap S_2 = \emptyset$ (possibly $\theta_2^+ = \theta_2^- + 2\pi$). We solve $\theta'_2 = \eta_2(\theta_2)$ on (θ_2^-, θ_2^+) , and let $r_2(\theta) = \exp(\int \rho_2/\eta_2 d\theta)$. As before we look at $A_2(\theta_2) = r_2(\theta_2)(\rho_1 - \rho_2)(\theta_2)$. Since A_2 is non-increasing (same estimate) and $r_2(\theta_2(\tau)) \rightarrow 0$ for $\tau \rightarrow -\infty$ we deduce that $\rho_1 \leq \rho_2$ on (θ_2^-, θ_2^+) . By using this argument for all possible intervals of this form, as well as continuity at S_2 , we finally see that $\rho_1 \leq \rho_2$ for all angles, and then by symmetry that $\rho_1 = \rho_2$. \square

12.2.2. Uniqueness of the lower singular cycle.

Proposition 12.11. *Suppose $V = 0$, Condition 12.1 and (2.10) for all $E \in I$. Then there is at most one energy in I for which a lower singular cycle exists, and this solution is uniquely determined.*

Proof. We define $b_1(\theta) = b(-\theta)$. The transformation $\rho(\theta) \rightarrow -\rho(-\theta)$ turns a lower singular cycle for b to an upper singular cycle for b_1 , cf. [CHS, Section 9]. Hence we can invoke Proposition 12.10 with $b \rightarrow b_1$. \square

12.3. Incoming spirals. We work in the remaining part of this section under the conditions of Subsection 12.1 (i.e. (2.9) and (2.10) are not used). Let us denote the set \mathcal{E} in (12.4) by \mathcal{E}^+ , and introduce

$$\mathcal{E}^- = \left\{ E > \max V \mid \text{a } 2\pi\text{-periodic solution } \rho_E \text{ exists,} \right. \\ \left. 2(E - V) > \rho_E^2, \int_0^{2\pi} \frac{\rho_E}{\sqrt{2(E - V) - \rho_E^2}} d\theta < 0 \right\}.$$

Here ρ_E solves (12.5) (as before).

The transformation $\rho(\theta) \rightarrow -\rho(-\theta)$ turns a solution ρ_E corresponding to $E \in \mathcal{E}^-$ to a solution to (12.5) with $b(\theta) \rightarrow b_1(\theta) = b(-\theta)$ and $V(\theta) \rightarrow V_1(\theta) = V(-\theta)$ but reverses the sign of the integral and hence $E \in \mathcal{E}^+ = \mathcal{E}^+(b_1, V_1)$, and vice versa. We previously studied \mathcal{E}^+ ; whence we may deduce for instance that \mathcal{E}^- is an interval $\mathcal{E}^- = (E_d^-, E_e^-)$ if $V = 0$, possibly empty.

In the case $b < 0$ we showed in [CHS] that $\mathcal{E}^+ = \mathcal{E}^- \neq \emptyset$.

For $a \in (-\sqrt{2E}, \sqrt{2E})$ and $E > 0$ we consider the initial value problem

$$\begin{cases} \rho_E' = b + \eta E; & \eta E = \sqrt{2(E - V) - \rho_E^2} > 0 \\ \rho_E(\theta = 0) = a \end{cases}. \quad (12.23)$$

If there is a solution on the interval $[0, 2\pi]$ we denote it by $\rho = \rho(\theta, a, E)$. Clearly ρ extends to a 2π -periodic solution if $\rho(2\pi, a, E) = a$.

Lemma 12.12. *Assume there exists a periodic solution ρ_d to (12.23) at an energy $E = E_d$, such that*

$$\int_0^{2\pi} \frac{\rho_d}{\sqrt{2(E - V) - \rho_d^2}} d\theta = 0.$$

Then \mathcal{E}^+ and \mathcal{E}^- are not empty, and $E_d = E_d^+ = E_d^-$. In particular for $V = 0$, if the set $\mathcal{E}^+ \neq \emptyset$ then $\mathcal{E}^- \neq \emptyset$ (and vice versa).

Proof. Let us introduce $a_d = \rho_d(\theta = 0)$. We look at the equation $f(a, E) := \rho(2\pi, a, E) - a = 0$ for $|a - a_d|$ and $|E - E_d|$ small. By ODE techniques we find that f is smooth in its variables. We compute

$$\partial_a f(a, E) = e^{-\int_0^{2\pi} \frac{\rho}{\eta} d\theta'} - 1; \quad \eta = \sqrt{2(E - V) - \rho^2}, \quad (12.24)$$

$$\partial_a^2 f(a, E) = -e^{-\int_0^{2\pi} \frac{\rho}{\eta} d\theta'} \int_0^{2\pi} \frac{2(E - V)}{\eta^3} e^{-\int_0^\theta \frac{\rho}{\eta} d\theta'} d\theta. \quad (12.25)$$

Next we evaluate (12.24) and (12.25) at $(a, E) = (a_d, E_d)$ and conclude that as a function of a , f vanishes to second order at (a_d, E_d) .

Another straightforward computation gives:

$$\begin{aligned} \partial_E f(a, E) &= e^{-\int_0^{2\pi} \frac{\rho}{\eta} d\theta'} \int_0^{2\pi} \eta^{-1} e^{\int_0^\theta \frac{\rho}{\eta} d\theta'} d\theta \\ \partial_E f(a_d, E_d) &= \int_0^{2\pi} \eta_d^{-1} e^{\int_0^\theta \frac{\rho_d}{\eta_d} d\theta'} d\theta > 0. \end{aligned} \quad (12.26)$$

Using the third order Taylor expansion for $f(\cdot, E)$ in a neighborhood of (a_d, E_d) , we have that $f(a, E) = 0$ iff

$$\begin{aligned} f(a, E) - f(a, E_d) & \\ &= -\frac{(a - a_d)^2}{2}(\partial_a^2 f)(a_d, E_d) - \int_{a_d}^a dx \int_{a_d}^x dy \int_{a_d}^y dt (\partial_a^3 f)(t, E_d). \end{aligned} \quad (12.27)$$

Recall that $(\partial_a^2 f)(a_d, E_d) < 0$, $(\partial_E f)(a_d, E_d) > 0$, and $f(a_d, E) - f(a_d, E_d) > 0$ for $E > E_d$ close to E_d . Now choose $\epsilon > 0$ and $\delta > 0$ small enough and define (here $E_d \leq E \leq E_d + \delta$)

$$\begin{aligned} F_{\pm} : [a_d - \epsilon, a_d + \epsilon] &\rightarrow \mathbb{R}, \\ F_{\pm}(a) &= a_d \\ &\pm \sqrt{\frac{f(a, E_d) - f(a, E)}{\frac{1}{2}(\partial_a^2 f)(a_d, E_d) + \frac{1}{(a - a_d)^2} \int_{a_d}^a dx \int_{a_d}^x dy \int_{a_d}^y dt (\partial_a^3 f)(t, E_d)}}. \end{aligned} \quad (12.28)$$

If δ is chosen small enough, we see that F_{\pm} leave their domains invariant, and moreover, they are contractions; more precisely, one can prove the estimate

$$\sup_{a \in [a_d - \epsilon, a_d + \epsilon]} |F'_{\pm}(a)| \leq C\sqrt{E - E_d}.$$

The two fixed points will in fact be the two possible solutions of $f(a, E) = 0$ near (a_d, E_d) , with the extra condition $E > E_d$. Therefore:

$$a_{\pm}(E) = a_d \pm \sqrt{-(E - E_d) \frac{2(\partial_E f)(a_d, E_d)}{(\partial_a^2 f)(a_d, E_d)}} + \mathcal{O}(E - E_d). \quad (12.29)$$

Using the mean value theorem for $\partial_a f$ we obtain

$$\begin{aligned} (\partial_a f)(a_{\pm}(E), E) &= (\partial_a f)(a_{\pm}(E), E) - (\partial_a f)(a_d, E_d) \\ &= \mp \sqrt{-2(E - E_d)(\partial_E f)(a_d, E_d) \cdot (\partial_a^2 f)(a_d, E_d)} + \mathcal{O}(E - E_d). \end{aligned} \quad (12.30)$$

Then (12.24) gives that

$$\pm \int_0^{2\pi} \frac{\rho_{\pm}}{\eta_{\pm}} d\theta > 0$$

and we are done. \square

Lemma 12.13. *Suppose $E^+ \in \mathcal{E}^+$, $E^- \in \mathcal{E}^-$ and $E^+ \leq E^-$. Denoting the corresponding solutions by $\rho_{E^+}^+$ and $\rho_{E^-}^-$, respectively, we have*

$$\rho_{E^-}^-(\theta) \leq \rho_{E^+}^+(\theta) \text{ for all } \theta. \quad (12.31)$$

Proof. We modify the last part of the proof of Proposition 12.10 introducing $A(\theta) = r(\theta)(\rho^+ - \rho^-)(\theta)$ where $r = \exp(\int \rho^-/\eta^- d\theta)$ and $\theta' = \eta^-(\theta)$. As before we verify that $A' \leq 0$. Whence, since $r \rightarrow 0$ for $\tau \rightarrow +\infty$, $\rho^- \leq \rho^+$. \square

12.4. Completeness in \mathcal{E}^+ , classical case. The purpose of this subsection is to outline a proof of asymptotic completeness in the set $\mathcal{E} = \mathcal{E}^+$, cf. Remark 12.9, that is generalizable to quantum mechanics. The one of Proposition 12.2 is not generalizable since it relies on topological arguments.

Fix $E \in \mathcal{E}$. We shall prove classical completeness for orbits with energy E . By a *scattering orbit* we mean below a classical orbit moving to infinity in configuration space, that is $r(t) = |x(t)| \rightarrow \infty$ for $t \rightarrow \infty$ ($= +\infty$). Since the case where $\mathcal{F}^+(E) = \emptyset$ essentially was treated in [CHS] let us assume that $\mathcal{F}^+(E) \neq \emptyset$.

We define $E_{\text{crt}} \in \mathcal{E}$ as the smallest energy $E_{\text{crt}} = E' > E$ for which the equation

$$\rho_{E'}(\theta) = \sqrt{2(E - V(\theta))} \quad (12.32)$$

has a solution $\theta \in \mathbb{T}$. Due to Proposition 12.7 this is a good definition. Let T_{crt} denote the set of all solutions $\theta \in \mathbb{T}$ for this particular energy. From the definition of E_{crt} and (12.5) it follows that

$$b + \eta_{\text{crt}} = -\frac{V'}{\rho_{E_{\text{crt}}}} \text{ at all angles } \theta \in T_{\text{crt}}. \quad (12.33)$$

Whence

$$\epsilon_1 := -\max_{T_{\text{crt}}} \left\{ b + \frac{V'}{\rho_{E_{\text{crt}}}} \right\} = \min_{\theta \in T_{\text{crt}}} \eta_{E_{\text{crt}}}(\theta) > 0. \quad (12.34)$$

In particular $T_{\text{crt}} \cap \mathcal{F}_1^+(E) = \emptyset$.

Let $T_\kappa = \{\theta | \rho_{E_{\text{crt}}}(\theta) > \sqrt{2(E - V(\theta))} - \kappa\}$; $\kappa > 0$. By continuity, for κ taken small enough

$$b(\theta) + \frac{V'}{\sqrt{2(E - V(\theta))}} \leq -\frac{\epsilon_1}{2} \text{ for all } \theta \in T_\kappa. \quad (12.35)$$

Proposition 12.14. *Let E, E_{crt} be given as above. Fix $E_1 \in (E, E_{\text{crt}})$. Then there exists $\epsilon > 0$ small enough such that the following bounds hold for all scattering orbits with energy E :*

- 1) $\int_1^\infty F(\rho - \rho_{E_{\text{crt}}} < \epsilon) F(\rho - \rho_{E_1} > \epsilon) r^{-1} dt < \infty$.
- 2) $F(\rho - \rho_{E_{\text{crt}}} < \epsilon) F(\rho - \rho_{E_1} > \epsilon) \rightarrow 0$ for $t \rightarrow \infty$.

Proof. As in [CHS] we introduce the observable $B_{E'} = \rho - \rho_{E'}(\theta)$ (for any $E' \in \mathcal{E}$) and compute

$$\mathbf{d}B_{E'} = (\eta - \eta_{E'}) \frac{\eta}{r}. \quad (12.36)$$

For $E' < E$ in \mathcal{E} (12.36) leads to the Mourre estimate

$$\mathbf{d}\{rB_{E'}\} \geq \delta \text{ for some } \delta > 0. \quad (12.37)$$

In particular we learn from (12.37) that for any given $E' < E$ in \mathcal{E} , $\rho(t) \geq \rho_{E'}(\theta(t)) \geq -\sqrt{2(E' - V(\theta(t)))}$ for all t large enough.

Using the Mourre estimate and a maximal velocity bound, cf. (7.11), we see that the factor r^{-1} in 1) may be replaced by t^{-1} . Hence the statement 2) follows from the subsequence argument, cf. the proof of Proposition 2.3 (or for example [CHS]), and 1).

If $\eta \neq -\eta_{E'}$ we may rewrite (12.36) as

$$\mathbf{d}B_{E'} = \frac{2(E - E')}{\eta + \eta_{E'}} \frac{\eta}{r} - \frac{\rho + \rho_{E'}}{\eta + \eta_{E'}} \frac{\eta}{r} B_{E'} = T_1 + T_2. \quad (12.38)$$

With these preliminaries we can start proving 1).

Step I We shall prove the bound

$$\int_1^\infty F(B_{E_{\text{crt}}} < \epsilon) F(-\eta > \epsilon') r^{-1} dt < \infty; \quad \epsilon' > 0. \quad (12.39)$$

For $E' \in (E_d, E_{\text{crt}}] \cap \mathcal{E}$ consider

$$\Phi = F(B_{E'} < 2\epsilon) F(-\eta > \epsilon') = F_1 F_2.$$

In the arguments below we assume that $\epsilon' > 0$ is ‘‘small’’. We compute using (12.36)

$$\mathbf{d}\Phi = F'(B_{E'} < 2\epsilon) (\eta - \eta_{E'}) F_2 \frac{\eta}{r} + F_1 F'(-\eta > \epsilon') \frac{(b + \eta)\rho + V'}{r}, \quad (12.40)$$

leading to

$$\mathbf{d}\Phi \leq F'(B_{E'} < 2\epsilon) F_2 \frac{\epsilon' \min \eta_{E'}}{2r}. \quad (12.41)$$

Here we used that the second term in (12.40) is non-positive due to the fact that on the support of $F_1 F'(-\eta > \epsilon')$, ρ is positive, and η and $b\rho + V'$ are negative, cf. (12.35).

Clearly (12.41) yields

$$\int_1^\infty |F'(B_{E'} < 2\epsilon)|F(-\eta > \epsilon')r^{-1} dt < \infty. \quad (12.42)$$

Taking the freedom of varying $E' \in (E_d, E_{\text{crt}}] \cap \mathcal{E}$ and the monotonicity in $\rho_{E'}$ into account we conclude (12.39) from (12.42), cf. the proof of [CHS, Proposition 3.5].

Step II For any sufficiently small $\epsilon > 0$ we shall prove the bound

$$\int_1^\infty F(B_{E_{\text{crt}}} < \epsilon)F(B_{E_1} > \epsilon)F(\eta > \epsilon')r^{-1} dt < \infty; \quad \epsilon' > 0. \quad (12.43)$$

Consider for $E' \in [E_1, E_{\text{crt}}]$ the observable

$$\Phi = F(B_{E'} < \bar{\epsilon})F(\eta > \epsilon') = F_1 F_2; \quad \bar{\epsilon} \in [\epsilon/2, 2\epsilon].$$

As before we may assume that $\epsilon' > 0$ is “small”. We compute using (12.38)

$$\mathbf{d}\Phi = F'(B_{E'} < \bar{\epsilon})(T_1 + T_2)F_2 - F_1 F'(\eta > \epsilon') \frac{(b + \eta)\rho + V'}{r}, \quad (12.44)$$

Now we may bound the term with T_1 from below as

$$F'(B_{E'} < \bar{\epsilon})T_1 F_2 \geq |F'(B_{E'} < \bar{\epsilon})|F_2 \frac{c(E' - E)\eta}{r} \quad (12.45)$$

for some $c > 0$. For the term with T_2 we have for some $C > 0$

$$F'(B_{E'} < \bar{\epsilon})T_2 F_2 \geq -|F'(B_{E'} < \bar{\epsilon})|F_2 \frac{C\bar{\epsilon}\eta}{r}. \quad (12.46)$$

Finally the last term on the right hand side of (12.44) is non-negative, cf. (12.35).

We conclude that for $\bar{\epsilon} \sim \epsilon \ll (E_1 - E)$ there is a $\delta > 0$ such that

$$\mathbf{d}\Phi \geq |F'(B_{E'} < \bar{\epsilon})|F_2 \frac{\delta\epsilon'}{r}. \quad (12.47)$$

From this we conclude (12.43) by varying $E' \in [E_1, E_{\text{crt}}]$ and $\bar{\epsilon} \in [\epsilon/2, 2\epsilon]$.

Step III By combining (12.39) and (12.43) with energy conservation we obtain that for all sufficiently small $\bar{\epsilon}, \epsilon > 0$

$$\int_1^\infty F(B_{E_{\text{crt}}} + \epsilon + \bar{\epsilon} < \epsilon)F(B_{E_1} > \epsilon)r^{-1} dt < \infty. \quad (12.48)$$

Step IV It remains to establish an estimate for the region $-2\epsilon < B_{E_{\text{crt}}} < \epsilon$. For that we introduce the observable

$$\Phi = \eta F(B_{E_{\text{crt}}} < \epsilon)F(B_{E_{\text{crt}}} + 2\epsilon > \epsilon) = \eta F_1 F_2.$$

We compute

$$\mathbf{d}\Phi = -\frac{(b + \eta)\rho + V'}{r} F_1 F_2 + \eta F_2 F'(B_{E_{\text{crt}}} < \epsilon) \mathbf{d}B_{E_{\text{crt}}} + \eta F_1 \mathbf{d}F_2. \quad (12.49)$$

By Step III the last term is integrable. We insert for each of the other two terms $I = F(-\eta > \epsilon') + F(\eta > \epsilon') + R$. The contributions from $F(-\eta > \epsilon')$ and $F(\eta > \epsilon')$ are integrable due to Step I and II (with $\epsilon \rightarrow 2\epsilon$). But for the contributions from R we use (12.38) to conclude that for some positive δ

$$\begin{aligned} & -\frac{(b + \eta)\rho + V'}{r} F_1 F_2 R + \eta F'(B_{E_{\text{crt}}} < \epsilon)(T_1 + T_2)F_2 R \\ & \geq \frac{\delta}{r} F_1 F_2 R + 0 = \frac{\delta}{r} F_1 F_2 R; \end{aligned} \quad (12.50)$$

here we used again (12.35) and the arguments in Step II. By replacing $R \rightarrow I$ (by the same arguments as before) we conclude that

$$\mathbf{d}\Phi \geq F_1 F_2 \frac{\delta}{r} + \text{integrable terms.} \quad (12.51)$$

From (12.51) we obtain

$$\int_1^\infty F(B_{E_{\text{crt}}} < \epsilon) F(B_{E_{\text{crt}}} + 2\epsilon > \epsilon) r^{-1} dt < \infty. \quad (12.52)$$

Clearly 1) follows from the bounds (12.48) and (12.52). \square

Corollary 12.15. *For all scattering orbits with energy $E \in \mathcal{E}$ one of the following two possibilities occur:*

- (i) $\lim_{t \rightarrow \infty} |\rho(t) - \rho_E(\theta(t))| = \lim_{t \rightarrow \infty} |\eta(t) - \eta_E(\theta(t))| = 0.$
- (ii) $\lim_{t \rightarrow \infty} \left\{ \eta^2(t) + \left(b(\theta(t))\rho(t) + V'(\theta(t)) \right)^2 \right\} = 0.$

Proof. From Proposition 12.14 we learn that either

- 1) $\lim_{t \rightarrow \infty} |\rho(t) - \rho_E(\theta(t))| = 0,$ or
- 2) $\liminf_{t \rightarrow \infty} (\rho(t) - \rho_{E_{\text{crt}}}(\theta(t))) > 0.$

In the case of 1) we have (i). In the case of 2) we change $b \rightarrow \bar{b}$ in a small neighborhood of some $\theta_{\text{crt}} \in T_{\text{crt}}$ such that $\int_0^{2\pi} \bar{b} d\theta = 0.$ We can then proceed as in the proof of Proposition 2.3 with $b \rightarrow \bar{b}$ using Proposition 12.14 to treat errors involving the expression $b - \bar{b}.$ \square

13. QUANTUM MECHANICS IN THE GENERAL SETTING

We shall outline how the methods used in the zero flux case to prove asymptotic completeness may be modified using now the classical theory of Section 12.

In addition to Condition 12.1 we shall in this section impose the following condition.

Condition 13.1 (No large oscillation). For all orbits in the reduced phase space $\mathbb{T}_E^* = \{z \in \mathbb{T}^* \mid h(z) = E\}$ with energy $E \in I,$

$$\inf_{\tau_2 > \tau_1} \left\{ \left(\sqrt{2(E - V(\theta(\tau_2)))} + \rho(\tau_2) \right) + \left(\sqrt{2(E - V(\theta(\tau_1)))} - \rho(\tau_1) \right) \right\} > 0. \quad (13.1)$$

We remark that Condition 13.1 is fulfilled 1) for $E > E_d^+$ in the case where $\mathcal{E}^+ \neq \emptyset$ (proved by checking that $e^{\int \rho} (\rho - \rho_{E'}^+)$ is increasing in τ for $E' < E$ in $\mathcal{E}^+),$ 2) in general for all high enough energies or if $b = 0$ (proved by using that $e^{\int \rho}$ is increasing in $\tau).$

Clearly Condition 13.1 excludes Proposition 12.2 i). Consequently, under the conditions of Lemma 12.12, the condition (13.1) is false for all orbits with energy $E < E_d^+ = E_d^-$ such that $\mathcal{F}(E) = \emptyset.$ Notice that in this case indeed Proposition 12.2 i) occurs. An example is the case, $V = 0$ and $b < 0,$ treated in [CHS]. We remark that Condition 13.1 implies Condition 2.4.

Due to the discreteness of $\mathcal{E}_{\text{exc}},$ cf. Appendix A, Condition 13.1 may be relaxed: For this section it would suffice to impose the condition that Proposition 12.2 i) as well as its negative time version, cf. Remark 12.3, do not occur.

Condition 13.2 (No singular cycles). The interval I does not contain an energy at which there exists a singular cycle.

Notice that by Propositions 12.10 and 12.11 singular cycles may occur at most at two energies in the case $V = 0.$

13.1. A partial ordering. With Conditions 12.1, 13.1 and 13.2, and (2.9) and (2.10) at all $E \in I$, we can order the set $\mathcal{F}_{\text{sa}}^+(E)$ for any such E as follows: We write $z \prec \tilde{z}$ for $z, \tilde{z} \in \mathcal{F}_{\text{sa}}^+(E)$ if and only if, either $z = \tilde{z}$, or there exists a chain of orbits $\gamma_1, \dots, \gamma_n$, $n \in \mathbb{N}$, each orbit solving (2.6) and possessing energy E , such that

$$\lim_{\tau \rightarrow +\infty} \gamma_1(\tau) = \tilde{z}, \quad \lim_{\tau \rightarrow -\infty} \gamma_n(\tau) = z, \quad (13.2)$$

$$\lim_{\tau \rightarrow -\infty} \gamma_j(\tau) = \lim_{\tau \rightarrow +\infty} \gamma_{j+1}(\tau) \in \mathcal{F}_{\text{sa}}^+(E); \quad 1 \leq j < n \text{ (for } 1 < n). \quad (13.3)$$

We claim that this recipe is a partial ordering of $\mathcal{F}_{\text{sa}}^+(E)$: The only non-trivial property to check is anti-symmetry. So suppose $z \prec \tilde{z}$ and $\tilde{z} \prec z$, then we need to verify that $z = \tilde{z}$, which in turn amounts to showing that there are no loops. A loop is given by a chain $\gamma_1, \dots, \gamma_n$, $n \in \mathbb{N}$, such that the two limits in (13.2) are replaced by the same value $z = \tilde{z} \in \mathcal{F}_{\text{sa}}^+(E)$ (while (13.3) is left unchanged). Suppose first $n = 1$, so that we look at a homoclinic orbit γ . Since $\frac{d}{d\tau}(\rho - \tilde{b}) = \eta^2$ there exists $m \in \mathbb{N}$ such that

$$[\theta(\tau)]_{-\infty}^{\infty} = 2\pi m. \quad (13.4)$$

Now, due to Conditions 12.1 and 13.1, $\rho(\tau) > -\sqrt{2(E - V(\theta(\tau)))}$ for all τ . This means that $\gamma(\tau) \in \mathbb{T}^{*,+}(E)$ (see (12.2) for definition).

The topological structure of $\mathbb{T}^{*,+}(E)$ is an annulus; thus γ is a non-self-intersecting closed orbit in an annulus. The equation of motion for the angle is $\theta' = \eta$. Under these circumstances the only possibility for angular increment along the orbit is $m = 1$ in (13.4), cf. [F, Proposition 5.20], and $\eta(\tau) \geq 0$ for all τ . (Notice that $\{\eta = 0\}$ divides $\mathbb{T}^{*,+}(E)$ into two separate annuli with opposite direction of flow, and that $\eta(\tau) \leq 0$ is excluded by the flux-condition $\tilde{b}(2\pi) \leq 0$.) The existence of such orbit is not consistent with Condition 13.2.

In the case $n > 1$ the total angular increment for the loop is again given by $2\pi m$ for some $m \in \mathbb{N}$, and we may argue as above. It does not exist.

The partial ordering defines the *order* of any given $\tilde{z} \in \mathcal{F}_{\text{sa}}^+(E)$ as the largest possible n for which there exists $z \in \mathcal{F}_{\text{sa}}^+(E)$ and a chain of orbits $\gamma_1, \dots, \gamma_n$ such that (13.2) and (13.3) hold. If \tilde{z} is minimal the order is by definition $n = 0$.

We order $\mathcal{F}_{\text{sa}}^-(E)$ in the same way as done above for $\mathcal{F}_{\text{sa}}^+(E)$ (replacing $\mathcal{F}_{\text{sa}}^+(E)$ in (13.3) by $\mathcal{F}_{\text{sa}}^-(E)$). The arguments for anti-symmetry are similar. Instead of (12.2) we consider the set $\mathbb{T}^{*,-}(E)$ of (12.3).

13.2. The case $\mathcal{E}^+ \cup \mathcal{E}^- = \emptyset$. Suppose the conditions of Subsection 13.1. Using Proposition 12.2 and the orderings of Subsection 13.1 we shall outline a proof of the limiting absorption principle and asymptotic completeness in the case where $\mathcal{E}^+ \cup \mathcal{E}^- = \emptyset$ along the lines of Sections 3–9 and 11. Notice that Subsection 4.1 generalizes verbum verbatim, and that Proposition 12.2 ii) (and its negative time version, cf. Remark 12.3) occurs for all orbits with energy $E \in I$.

13.2.1. Bounds at $\mathcal{F}_{\text{sa}}^-$. We shall modify Section 5 using now the more refined ordering at $\mathcal{F}_{\text{sa}}^-(E)$ introduced above.

So suppose first that $z(E) \in \mathcal{F}_{\text{sa}}^-(E)$ is minimal at energy $E = E_0 \in I$ in the sense of Subsection 13.1. Then we use the same constructions as in Section 5: We aim again at proving (5.1) by using the quantizations of the symbols in (4.1) with f supported in an interval $[E_0 - \delta, E_0 + \delta]$ with $\delta \ll \epsilon$. We define for $\epsilon > 0$

$$K = \left\{ z \in \mathbb{T}_-^* \mid \frac{\epsilon}{4} \leq l_{E_0} \leq 2\epsilon, \quad h = E_0, \quad \text{dist}(z, M_{E_0}^u) \geq \frac{\epsilon}{2} \right\}.$$

If ϵ is small enough, then for all $z \in K$ we cannot have $\phi_\tau(z) \rightarrow \tilde{z} \in \mathcal{F}_{\text{sa}}^-$ for $\tau \rightarrow -\infty$, due to the fact that $z(E_0)$ is minimal. Here we used Proposition 12.2. In fact, since Proposition 12.2 ii) and its negative time version are valid, in the far past $\phi_\tau(z)$ will be close to $\mathcal{F}_{\text{so}}^-(E_0)$. Notice that also Condition 2.4 is used here

(recall that it is a consequence of Condition 13.1). We choose $\tau_0 \ll -1$ such that we have a good wave front set bound of an open neighborhood $U_{\tau_0} \subseteq \mathbb{T}^*$ of the compact set $\phi_{\tau_0}(K)$. Define $U_1 = \phi_{-\tau_0}(U_{\tau_0})$,

$$U_2^\delta = \{z \in \mathbb{T}^* \mid \frac{\epsilon}{4} < l_h < 2\epsilon, |h - E_0| < 2\delta, \text{dist}(z, M_h^u) < \epsilon\}; \delta > 0,$$

and

$$K^\delta = \{z \in \mathbb{T}^* \mid \frac{\epsilon}{2} \leq l_h \leq \epsilon, |h - E_0| \leq \delta, \text{dist}(z, M_h^u) \geq \epsilon\}; \delta \geq 0.$$

Due to Lemma 5.1

$$\tilde{K} := \{z \in \mathbb{T}^* \mid \frac{\epsilon}{2} \leq l_h \leq \epsilon, |h - E_0| \leq \delta\} \subseteq U_1 \cup U_2^\delta.$$

Next pick non-negative $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^*)$ subordinate to this covering, that is $\text{supp } \psi_1 \subseteq U_1$, $\text{supp } \psi_2 \subseteq U_2^\delta$ and $\psi_1 + \psi_2 = 1$ on \tilde{K} .

We decompose $F'(l_h < \epsilon)f(h) = (\psi_1 + \psi_2)F'(l_h < \epsilon)f(h)$. The propagation of singularity theorem applies to the contribution from the term involving ψ_1 . Again the contribution from the term involving ψ_2 has the right sign. We conclude (5.1) if the order of $z(E_0)$ is zero.

We repeat the analysis at elements $z \in \mathcal{F}_{\text{sa}}^-(E)$ of positive order measured at $E = E_0$, proving “good” bounds there too. As in Subsection 5 this is done by grouping the elements according to their order and treat recursively the groups one by one in increasing order. Finally we vary $E_0 \in I$ and conclude (5.1) for any $f \in C_0^\infty(I)$.

13.2.2. Bounds at $\mathcal{F}_{\text{sa}}^+$. We may mimic the analysis of the previous subsection at elements $z \in \mathcal{F}_{\text{sa}}^+(E)$ starting again at elements of order zero measured at $E = E_0$ and treating elements of higher order recursively. We use the observable of Subsection 4.3, and we may obtain (4.11), in fact without the last term to the right. Notice at this point that we do not have (4.10); we use at each step a stronger local version derived by propagation of the established bounds at \mathcal{F}^- and at the saddles in $\mathcal{F}_{\text{sa}}^+(E)$ of lower order.

13.2.3. Bounds at $\mathcal{F}_{\text{si}}^+$ and away from \mathcal{F} . We may obtain (4.11) at $\mathcal{F}_{\text{si}}^+$ by using the propagation of singularity result. Similarly we may prove (4.11) away from \mathcal{F} .

However we shall also need a stronger type of bound like (4.10) for the limiting absorption principle, cf. (6.7). The idea of the proof is to look at the construction (4.1) with $t = \frac{1}{2}$ rather than $t > \frac{1}{2}$ as above, first at points in $\mathcal{F}_{\text{sa}}^+$ (treated in increasing order) and then at points in $\mathcal{F}_{\text{si}}^+$. Whence by repeating the analysis above we may obtain $WF^{-1/2}$ -bounds in deleted neighborhoods of \mathcal{F}^+ . Using again the propagation of singularity result and the (strong) bounds at \mathcal{F}^- we consequently obtain the bound

$$\begin{aligned} \|\text{Op}^w(\chi)u\|_{-1/2}^2 &\leq C_1\|u\|_{-1}^2 + \sigma\|u\|_{-t}^2 + C_2\sigma^{-1}\|v\|_t^2; \\ \text{supp } \chi &\subset \mathbb{T}^* \setminus \mathcal{F}^+, \quad \sigma > 0, \quad t \in (1/2, 1). \end{aligned} \quad (13.5)$$

13.2.4. Limiting absorption principle. We may now mimic Section 6 to show the analogue of Theorem 6.4.

13.2.5. Preliminary estimates. Using (6.13) and (13.5) we obtain the following bound away from eigenvalues:

$$WF^{-\frac{1}{2}}(R(E + i0)v) \subseteq \mathcal{F}^+ \text{ for all } v \in L^{2,t}; \quad t > 1/2. \quad (13.6)$$

Given (13.6) we may obtain (by the Fourier transform) a version of (7.3); to the left we have the expectation of $\text{Op}^w(\chi)$ with χ as in (13.5), however $\|\psi\|$ to the right needs to be replaced by $\|\psi\|_t$ with $t > \frac{1}{2}$. Since (7.3) is used to derive the integral estimates (7.5) and (7.6) the right hand side of those needs to be changed similarly.

Whence we have a slightly weaker version of Lemma 7.1. We may verify Lemma 7.2 as before for $\psi \in L^{2,t}$ and hence for $\psi \in L^2$.

13.2.6. *Decomposition into channels formula.* As for Section 8 we need to proceed differently; it is not possible to show the existence of the projections P_j along the same line. This is due to the fact that we only have integral bounds with $\|\psi\|_t$ appearing to the right.

Instead we may start by defining P_j at the stable fixed points. We define in this case, in terms of the operator $L_1(t) = \text{Op}^w(l_1(t))$ with $l_1(t)$ specified in (11.6),

$$P_j f(H)\psi = \lim_{t \rightarrow \infty} e^{itH} L_1(t) e^{-itH} f(H)\psi, \quad (13.7)$$

for $\psi \in L^{2,t}$; $t > \frac{1}{2}$. Here we impose (11.5); consequently due to Lemma 7.2 and the proof of Lemma 8.3 the right hand side is independent of the ϵ 's. By definition P_j is the smallest orthogonal projection obeying (13.7). We notice that indeed the limit exists, cf. (11.4) and (11.7)–(11.12) (notice that $\|\psi\|$ appears in the integral bounds). Whence (by Lemma 7.2 and the proof of Lemma 8.3) P_j is well-defined.

We are left with proving the formula

$$1_I(H) = P_I := \sum_{z_j(\cdot) \in \mathcal{F}_{\text{st}}^+} P_j. \quad (13.8)$$

This may be done by looking at the evolution of states of the form

$$\psi_{\text{exc}} = (1_I(H) - P_I) f(H)\psi; \quad \psi \in L^{2,t}, \quad t > \frac{1}{2}.$$

We may decompose

$$\psi_{\text{exc}}(t) = e^{-itH} \psi_{\text{exc}} \approx \sum_{z_j(\cdot) \in \mathcal{F}_{\text{sa}}^+} \text{Op}^w(\chi_j) e^{-itH} f(H)\psi, \quad (13.9)$$

where χ_j is a symbol localized near $z_j(\cdot)$.

At this stage we can show that indeed

$$\lim_{t \rightarrow \infty} \|e^{itH} \text{Op}^w(\chi_j) e^{-itH} f(H)\psi\| = 0$$

by invoking [HS2, Theorem 1.2].

13.2.7. *Completeness.* Given (13.8) we may proceed as in the case of zero flux in Section 11, in particular we may obtain analogues of Theorems 11.1 and 11.2.

13.3. **The case $\mathcal{E}^+ \neq \emptyset$.** Suppose the conditions of Subsection 13.1. In the case $\mathcal{E}^+ \neq \emptyset$ we shall discuss quantum completeness above E_d^+ . The limiting absorption principle follows from quantizing (12.37), cf. [CHS]. For completeness in \mathcal{E}^+ we refer to Subsection 12.4 and [CHS]. Notice that the arguments given in Subsection 12.4 generalize to quantum mechanics. Consequently the outcome is analogues of Theorems 11.1 and 11.2 for the part of the wave packets at fixed points and spiraling behaviour as in [CHS] for another part, cf. (1.12) and (1.13).

So we focus on the case where $I \cap \overline{\mathcal{E}^+} = \emptyset$. (Notice that this amounts to looking at $I \subseteq (E_e^+, \infty)$ if $V = 0$.)

Suppose first that also $I \cap \mathcal{E}^- = \emptyset$. Then again Proposition 12.2 ii) and its negative time version hold, and we obtain the wave front set bound (13.6) and completeness as before.

If $I \cap \mathcal{E}^- \neq \emptyset$ the incoming spiral may seem to cause problems in establishing (13.6). However the Mourre estimate gives a good bound of Au where $A = \text{Op}^w(F(r > 1)f(h)F(\rho - \rho_E^- < \epsilon))$ for some $\epsilon > 0$ and $u = R(E + i0)v$, due to the (classical) inequalities

$$\rho - \epsilon \geq \rho_{E'}^+ \geq \rho_E^-; \quad E' < E \text{ in } \mathcal{E}^+ \text{ and } \epsilon = \epsilon(E') > 0. \quad (13.10)$$

Here we assume that ρ belongs to an orbit with energy $E \in I$. The first inequality is due to the Mourre estimate while the second follows from Lemma 12.13. By a “good” bound we mean the statement $WF^{-s}(Au) = \emptyset$ for some $s < \frac{1}{2}$. Technically the implementation of (13.10) for this purpose may be done by considering the symbol

$$F(r > 1)f(h)(r(3\epsilon - \rho + \rho_{E'}^+))^{1-2s}F(\rho - \rho_{E'}^+ < 2\epsilon);$$

we skip the details. See [GIS] for a somewhat similar construction and procedure.

The outcome is again analogues of Theorems 11.1 and 11.2.

Remark 13.3. Quantum mechanics for energies below $\max V$ is easier to deal with, cf. Remark 12.5, and will not be discussed.

APPENDIX A. COLLAPSING AND EXCEPTIONAL CLASSICAL ORBITS

Under Condition 12.1 we shall study the collapsing orbits with energy $E \in I$ in the “torus” $\mathbb{T}_E^* = \{z \in \mathbb{T}^* \mid h(z) = E\}$. By definition a *collapsing orbit* is an integral curve of the system (2.6) for which the quantity $r = \exp(\int_0^\tau \rho \, d\tau) \rightarrow 0$ for $\tau \rightarrow +\infty$. We encountered such an example in Remark 2.6. The set of z 's in \mathbb{T}_E^* in the range of a collapsing orbit is denoted by $A_{\text{col}}(E)$.

We introduce a new independent variable ψ by

$$(\cos \psi, \sin \psi) = \frac{(\eta, \rho)}{\sqrt{2(E - V)}}, \quad (\text{A.1})$$

and compute

$$\psi' = \sqrt{2(E - V)} \cos \psi + b + \frac{V'}{\sqrt{2(E - V)}} \sin \psi =: F_2. \quad (\text{A.2})$$

Combining (A.2) and

$$\theta' = \sqrt{2(E - V)} \cos \psi =: F_1 \quad (\text{A.3})$$

leads to the following formula for the Jacobian of the flow $\phi_\tau = (\theta(\tau), \psi(\theta))$ for the vector field $F = (F_1, F_2)$ on \mathbb{T}_E^* ,

$$\exp\left(\int_0^\tau \nabla \cdot F \, d\tau\right) = \exp\left(\int_0^\tau -\rho \, d\tau\right) = r^{-1}. \quad (\text{A.4})$$

Hence for any measurable set $A \subseteq \mathbb{T}_E^*$

$$\int_{\phi_\tau(A)} d\theta d\psi = \int_A r(\tau; \theta, \psi)^{-1} d\theta d\psi. \quad (\text{A.5})$$

Lemma A.1. *The set $A_{\text{col}}(E) \subseteq \mathbb{T}_E^*$ has (relative) measure zero.*

Proof. We combine (A.5) and the Fatou Lemma. \square

Under more conditions the structure of $A_{\text{col}}(E)$ can be specified further. For example under the conditions of Proposition 2.3, $A_{\text{col}}(E)$ is the union of stable manifolds/curves at $\mathcal{F}^-(E)$. A less obvious example is provided by the following result.

Lemma A.2. *Suppose $\int_0^{2\pi} b \, d\theta < 0$ and $\mathcal{E}^- \neq \emptyset$. Then for any $E \in \mathcal{E}^-$ the associated orbit $\theta^- \rightarrow \gamma^-(\theta^-) = (\theta^-, \eta^-(\theta^-), \rho^-(\theta^-))$ is collapsing, and if $\gamma = (\theta, \eta, \rho)$ is another collapsing orbit with energy E then*

$$\rho(\tau) \leq \rho^-(\theta(\tau)). \quad (\text{A.6})$$

Suppose in addition that $\mathcal{F}^-(E) = \emptyset$ then

$$A_{\text{col}}(E) = \{\gamma^-(\theta^-) \mid \theta^- \in \mathbb{R}\}. \quad (\text{A.7})$$

Proof. As for γ^- we have $r = \exp(\int_0^\tau \rho \, d\tau) = \exp(\int_0^\tau \frac{\rho^-}{\eta^-} \, d\theta^-) \rightarrow 0$ for $\tau \rightarrow +\infty$.

Next we look at $A(\tau) = A = r(\rho^-(\theta) - \rho)$ defined in terms of ρ^- and the given collapsing orbit γ . We compute $A' \leq 0$, and since $A \rightarrow 0$ for $\tau \rightarrow +\infty$ we conclude that $A \geq 0$. Whence (A.6) holds.

Finally under the additional condition $\mathcal{F}^-(E) = \emptyset$ the function $f(\theta) = f = b\sqrt{2(E-V)} - V'$ has a definite sign. As in the proof of Proposition 12.2 we see that $\theta(\tau) \rightarrow +\infty$. We distinguish between the cases 1) $\eta(\tau_n) = 0$ along a sequence $\tau_n \rightarrow +\infty$, 2) $\eta > 0$ for all τ large enough.

We show that 1) does not occur: Since $\eta' = -b\rho - V'$ at any crossing given by $\eta = 0$ and the sign of η' must alternate for two consecutive crossings, we conclude from the fact that f has a definite sign that the sign of ρ cannot be negative for two consecutive crossings. In particular $\rho = \sqrt{2(E-V)}$ occurs, which contradicts (A.6).

In the case 2) we consider $A(\tau) = A = \{\exp(\int \frac{\rho^-}{\eta^-} \, d\theta)(\rho - \rho^-)\}(\theta^-)$ where $\theta^{-'} = \eta^-(\theta^-)$ and ρ is considered as a function θ . The latter is legitimate since $\theta = \int \eta \, d\tau$ increases monotonely to ∞ ; in particular we may consider τ as a function of the angle. Again we have $A' \leq 0$ and $A \rightarrow 0$ for $\tau \rightarrow +\infty$, whence $A \geq 0$. In combination with (A.6) we conclude that

$$\rho(\tau) = \rho^-(\theta(\tau)) \text{ for all large } \tau. \quad (\text{A.8})$$

From (A.8) we obtain (A.7) by noting that a similar relation holds for η and η^- (since $\eta > 0$ eventually). \square

We complete the appendix by showing discreteness of the exceptional set of energies \mathcal{E}_{exc} of Condition 2.4 (under Condition 12.1), cf. Remark 2.6.

Lemma A.3. *The set \mathcal{E}_{exc} is discrete in I .*

Proof. Suppose $E_1 \in \mathcal{E}_{\text{exc}}$. Let γ be a corresponding heteroclinic orbit and $z_1 = \gamma(0)$. All that needs to be checked is that γ splits at z_1 under perturbation with respect to variation of the energy. More precisely we may introduce a (fixed) transversal curve at z_1 and look at its intersection with the unstable orbit from one of the saddles (as a function of the energy-parameter) and similarly look at its intersection with the stable orbit from the other saddle. The homoclinic orbit splits if for all energies in a small deleted neighborhood of E_1 the two points of intersection are different. There is a criterion for splitting due to Melnikov [Meln]. Here we refer the reader to [C, Section 6.1]. It suffices to check that the ‘‘Melnikov integral’’ [C, (6.12)] (with $\lambda_j = E - E_1$) is nonzero. Due to (A.4) this amounts in our case to checking that

$$\int_{-\infty}^{\infty} r(2V'\omega^{-2} \cos \psi \sin \psi + b\omega^{-1} \cos \psi) \, d\tau \neq 0;$$

here $\omega = \sqrt{2(E-V)}$ and the integrand is evaluated at the unperturbed heteroclinic orbit $\tau \rightarrow \gamma(\tau)$.

This integral may be computed as follows: We express the last term as

$$b\omega^{-1} \cos \psi = \omega^{-2} \frac{d}{d\tau} \rho - \omega^{-2} \eta^2,$$

and treat the first term to the right after substitution by integrating by parts. Since $r \rightarrow 0$ for $|\tau| \rightarrow \infty$ boundary terms disappear. After a cancellation the integrand simplifies yielding the following expression

$$- \int_{-\infty}^{\infty} r\omega^{-2}(\eta^2 + \rho^2) \, d\tau = - \int_{-\infty}^{\infty} r \, d\tau$$

for the above integral. Obviously it is negative. \square

Remark A.4. For the case $V = 0$ one can construct examples of b 's for which $\mathcal{E}_{\text{exc}} \neq \emptyset$ as follows: Fix any non-constant b with $\int_0^{2\pi} b \, d\theta = 0$, and consider $b_\kappa = b + \kappa$ with $\kappa \in (-2 \max |b|, 0)$. Claim: There exist $\kappa_0 \in (-2 \max |b|, 0)$, $E_0 > 0$, and angles θ_1 and θ_2 such that upon replacing $b \rightarrow b_{\kappa_0}$ and $E \rightarrow E_0$ in (12.5) indeed the equation has a periodic solution $\rho = \rho_0$ obeying $\rho(\theta_1) = \sqrt{2E_0}$ and $\rho(\theta_2) = -\sqrt{2E_0}$. Given this claim we solve $\frac{d}{d\tau}\theta = \sqrt{2E_0 - \rho^2} =: \eta$ to obtain an exceptional orbit.

To outline a proof of the claim we introduce $f(E, \kappa) = \int_0^{2\pi} \frac{\rho_{E, \kappa}}{\eta_{E, \kappa}} \, d\theta$ for $\kappa < -\max |b|$ and sufficiently large E 's. We look at the condition $f(E, \kappa) = 1$, for example, starting from a fixed solution $\rho_{E, \kappa}$ and then trace the dependence of κ using the implicit function theorem. Notice here the properties Proposition 12.6 3), 6) and 8. We may in fact compute using Proposition 12.6 6), $E' = \frac{d}{d\kappa}E = -\partial_\kappa f / \partial_E f < 0$. Let us denote by κ_0 the right endpoint of the maximal κ -interval legitimate for this procedure. We may use the Arzela–Ascoli theorem to define a limiting solution $\rho = \lim_{\kappa \rightarrow \kappa_0^-} \rho_{E(\kappa), \kappa}$. If $\kappa_0 = 0$ we integrate (12.5) yielding $\int_0^{2\pi} \eta \, d\theta = 0$, which means that ρ is constant, and therefore in turn $b = 0$. By assumption this cannot be. Consequently, indeed $\kappa_0 < 0$.

It is an open problem whether $\mathcal{E}_{\text{exc}} \neq \emptyset$ may occur for the zero flux case.

APPENDIX B. A SIMILAR MODEL

Consider the symbol h on $(\mathbf{R}^2 \setminus \{0\}) \times \mathbf{R}^2$

$$h = h(x, \xi) = \frac{1}{2}g^{-1}\xi^2,$$

where the conformal (inverse) metric factor is specified in polar coordinates $x = (r \cos \theta, r \sin \theta)$ as $g^{-1} = e^f$; $f = f(\theta - c \ln r)$. We assume f is a given smooth non-constant 2π -periodic function and that $c > 0$. We introduce $v = (x_1 - cx_2)\partial_{x_1} + (cx_1 + x_2)\partial_{x_2} - c\xi_2\partial_{\xi_1} + c\xi_1\partial_{\xi_2}$. Computations show that v and the Hamiltonian vector field v_h fulfill the conditions of [HS2, Appendix A] along the positive orbit of v originating at $(r_0, 0; \rho_0, c\rho_0)$; here $\rho_0 = \sqrt{2E(1+c^2)^{-1}e^{-f_0}}$ where $f_0 = f(\theta_0)$ is given in terms of any $r_0 > 0$ satisfying the equation

$$-f'(\theta_0) = 2c(1+c^2)^{-1}; \quad \theta_0 = -c \ln r_0, \quad (\text{B.1})$$

and $E = h > 0$ is arbitrary. (Notice that there are at least two solutions to (B.1) for all small as well as for all large values of c .) The x -space part of the orbit (a geodesic) is the logarithmic spiral given by the equation $\theta - c \ln r = \theta_0$. We take $S \subset \{(x, \xi) | x_2 = 0\}$, cf. [HS2, Appendix A], and compute the eigenvalues for the linearized reduced flow to be given by

$$-\rho_0 \frac{1}{2} \left\{ 1 \pm \sqrt{1 - 2(1+c^2)^2 f_0''} \right\}; \quad f_0'' = f''(\theta_0). \quad (\text{B.2})$$

For $f_0'' < 0$ the family of fixed points consists of saddles. There are no resonances for “generic” values of c , and we also notice that taking $c \rightarrow 0$ in (B.1) and (B.2) yields the formulas for the corresponding homogeneous model (here the equations are considered to be equations in c and θ_0).

Finally, using the new angle $\tilde{\theta} = \theta - c \ln r$ one may conjugate to a homogeneous model. More precisely the relevant symplectic change of variables is induced (expressed here in terms of rectangular coordinates) by the map $x \rightarrow \tilde{x} = (x_1 g_1 + x_2 g_2, x_2 g_1 - x_1 g_2)$, where $g_1 = \cos(c \ln |x|)$ and $g_2 = \sin(c \ln |x|)$. One may check that $v \rightarrow \tilde{v} := \sum x_j \partial_{x_j}$, and that $h \rightarrow \tilde{h}$ given by

$$\tilde{h} = \frac{1}{2}e^{f(\theta)} \left(\{(c \sin \theta + \cos \theta)\xi_1 + (\sin \theta - c \cos \theta)\xi_2\}^2 + \{-\sin \theta \xi_1 + \cos \theta \xi_2\}^2 \right);$$

we changed notation back to the old one, $x = (r \cos \theta, r \sin \theta)$ for position and ξ for momentum.

Introducing dual polar variables ρ and l for r and θ , respectively, the expression for this Hamiltonian simplifies as

$$h = \frac{1}{2}e^{f(\theta)}((\rho - cl/r)^2 + (l/r)^2).$$

Let us introduce a new angle ψ by the relations

$$\begin{aligned} \rho &= \sqrt{2E(1+c^2)}e^{-f(\theta)/2} \cos \psi, \\ (1+c^2)l/r - c\rho &= \sqrt{2E(1+c^2)}e^{-f(\theta)/2} \sin \psi. \end{aligned}$$

The equations of motion are reduced in the variables θ and ψ :

$$\begin{cases} \frac{d}{d\tau}\theta = \sin \psi \\ \frac{d}{d\tau}\psi = -\frac{1}{2}g(\theta) \cos \psi - \frac{1}{1+c^2} \sin \psi \end{cases} \quad ; \quad (\text{B.3})$$

here $g(\theta) = f'(\theta) + 2c/(1+c^2)$ and τ represents a ‘‘new time’’.

Notice that the fixed points are given by $g(\theta) = 0$ and $\psi \in \pi\mathbb{Z}$, in agreement with (B.1).

To study (B.3) it is useful to observe that the observable $a_1 = \exp(-\frac{1}{2}\int_0^\theta g d\theta) \cos \psi$ obeys

$$\frac{d}{d\tau}a_1 = \exp\left(-\frac{1}{2}\int_0^\theta g d\theta\right) \frac{\sin^2 \psi}{1+c^2}. \quad (\text{B.4})$$

If $\sin \psi \neq 0$ for all τ we may consider ψ as a function of θ and look at

$$\frac{d}{d\theta}\psi = -\frac{1}{2}g(\theta) \cot \psi - \frac{1}{1+c^2}. \quad (\text{B.5})$$

Equipped with (B.4) and (B.5) we can prove the following analogue of Proposition 12.2. If $\psi(\theta)$ is a 2π -periodic solution to (B.5) with $\sin \psi(\theta) \neq 0$ for all θ then we say $\psi = \psi_p$ is *regular*. If $\psi(\theta)$ is a continuous 2π -periodic function solving (B.5) away from the zero set of g and obeying 1) $\sin \psi(\theta) = 0$ for at least one zero θ of g , and 2) either $\sin \psi(\theta) \geq 0$ or $\sin \psi(\theta) \leq 0$ (for all θ), then we call $\psi = \psi_p$ *singular*. We suppose that g has at most a finite number of zeros, all of which are non-degenerate.

Proposition B.1. *Let $\gamma = (\theta, \psi)$ be an arbitrary solution of (B.3). Then one of the following cases occurs:*

- i) *The set of fixed points \mathcal{F} for (B.3) is non-empty, and there exists $z \in \mathcal{F}$ such that $\gamma(\tau) \rightarrow z$ for $\tau \rightarrow +\infty$.*
- ii) *There exists a regular solution $\psi_p = \psi_r(\theta)$ to the equation (B.5), such that*

$$\lim_{\tau \rightarrow +\infty} |\psi(\tau) - \psi_r(\theta(\tau))| = 0. \quad (\text{B.6})$$

- iii) *There exists a singular solution $\psi_p = \psi_s(\theta)$ to the equation (B.5), such that*

$$\lim_{\tau \rightarrow +\infty} |\psi(\tau) - \psi_s(\theta(\tau))| = 0. \quad (\text{B.7})$$

Proof. Using (B.4) we have two possibilities, either 1) $\theta(\tau)$ stays bounded near $+\infty$, or 2) $\theta(\tau)$ is not bounded near $+\infty$. In the case of 1) we introduce the observable

$$\frac{d}{d\tau}a_2 = Ca_1 - \exp\left(-\frac{1}{2}\int_0^\theta g d\theta\right)g(\theta) \sin \psi; \quad C \gg 1. \quad (\text{B.8})$$

We may show that

$$\frac{d}{d\tau}a_2 \geq \frac{1}{4} \exp\left(-\frac{1}{2}\int_0^\theta g d\theta\right)(\sin^2 \psi + g(\theta)^2). \quad (\text{B.9})$$

Since $a_2(\tau)$ by assumption is bounded we obtain an integral estimate which in turn, as in the proof of Proposition 2.3, may be used to conclude that $\sin^2 \psi + g(\theta)^2 \rightarrow 0$ yielding Proposition B.1 i).

As for the case 2) we notice that if $\cos \psi(\tau_0) \geq 0$ for some τ_0 then the same relation holds for all $\tau \geq \tau_0$, cf. (B.4). Consequently, either $\cos \psi(\tau) < 0$ for all τ 's or $\cos \psi(\tau) \geq 0$ for all sufficiently large values of τ . Let us introduce the sets $\mathbb{T}^{*,\pm} = \mathbb{T}^2 \setminus \{\cos \psi \neq \mp 1\}$. Then, either $\gamma(\tau) \in \mathbb{T}^{*,+}$ or $\gamma(\tau) \in \mathbb{T}^{*,-}$ for all sufficiently large τ 's. Let us for convenience in the following assume $\gamma(\tau) \in \mathbb{T}^{*,+}$. We claim that eventually either $\sin \psi > 0$ or $\sin \psi < 0$. This statement follows from the assumption 2) and (B.3) using the topological structure of $\mathbb{T}^{*,+}$ (it is an annulus). Let us for convenience in the following assume $\sin \psi < 0$. We may write $\psi = \psi(\theta)$ and use (B.5). Next we consider the quantity $\Delta(\theta) = \psi(\theta - 2\pi) - \psi(\theta)$. If $\Delta(\theta) = 0$ we have a regular solution to (B.5). If not we can construct a solution as

$$\psi_p(\theta) = \lim_{n \rightarrow \infty} \psi(\theta - 2\pi n); \quad (\text{B.10})$$

notice that this sequence is either monotone increasing for all θ or monotone decreasing for all θ (depending on the sign of $\Delta(\theta)$). If the sequence is decreasing we have that $\psi_p(\theta) \geq -\pi/2$ and therefore that ψ_p is regular (this is due to the fact that a_1 is increasing). So let us look at the case where the sequence is increasing. It is readily seen (using this monotonicity) that if $\psi_p(\theta_0) = 0$ at some θ_0 then $g(\theta_0) = 0$. We can now prove that ψ_p is continuous, in fact absolutely continuous: Let $I = [\theta_1, \theta_2]$ be such that $g(\theta) \neq 0$ on either $[\theta_1, \theta_2]$ or $(\theta_1, \theta_2]$. Let us only consider the first case. We write

$$\begin{aligned} & \psi(\theta_2 - 2\pi n) - \psi(\theta_1 - 2\pi n) \\ &= - \int_{\theta_1}^{\theta_2} \left\{ \frac{1}{2} g(\theta) \cot \psi(\theta - 2\pi n) + (1 + c^2)^{-1} \right\} d\theta. \end{aligned} \quad (\text{B.11})$$

Using the monotone convergence theorem we may take $n \rightarrow \infty$ in (B.11) yielding

$$\psi_p(\theta_2) - \psi_p(\theta_1) = - \int_{\theta_1}^{\theta_2} h(\theta) d\theta, \quad (\text{B.12})$$

where $h(\theta) = \frac{1}{2} g(\theta) \cot \psi_p(\theta) + (1 + c^2)^{-1}$ for $\theta \neq \theta_2$. In particular h is integrable. By ‘‘gluing’’ these formulas together we obtain a representation of the form (B.12) without any restriction on the interval I . This shows absolute continuity of ψ_p . Furthermore we obtain from these arguments that the limit (B.10) is attained locally uniformly in θ . Using these facts we deduce the conclusion ii) or iii).

For the other cases (assuming 2)) we may argue similarly. \square

Proposition B.2. *For any solution ψ_p as considered in Proposition B.1 its range is an interval of length $< \pi/2$.*

Proof. As in the previous proof we concentrate on the case $\sin \psi(\theta) \leq 0$. It suffices to show that $\cos \psi > 0$. For that we introduce $a_1(\theta)$ by substituting $\psi(\theta)$ in the argument for ψ in the defining expression for a_1 . Differentiating using (B.5) yields

$$\frac{d}{d\theta} a_1 = \exp \left(-\frac{1}{2} \int_0^\theta g d\theta \right) \frac{\sin \psi}{1 + c^2} < 0. \quad (\text{B.13})$$

Upon integrating (B.13) (splitting the interval of integration at possible zeros of g to deal with singular solutions) we see that $a_1(\theta) < a_1(\theta - 2\pi)$, but this means that

$$\left(\exp \left(-\frac{1}{2} \int_{\theta-2\pi}^\theta g d\theta \right) - 1 \right) \cos \psi(\theta) < 0,$$

from which we see, using $g(\theta) = f'(\theta) + 2c/(1 + c^2)$, that indeed $\cos \psi(\theta) > 0$. \square

Corollary B.3. *Suppose $g(\theta) = f'(\theta) + 2c(1 + c^2)^{-1} \leq 0$ on an interval of length $(1 + c^2)\pi/2$, then only i) of Proposition B.1 occurs.*

Proof. We need to exclude the existence of solutions ψ_p as considered in Proposition B.1. Suppose $I = [\theta_1, \theta_2]$ is an interval of length $(1 + c^2)\pi/2$ on which $g(\theta) \leq 0$. Again we consider only the case $\sin \psi(\theta) \leq 0$. Upon integrating (B.5) we learn from the proof of Proposition B.2 that $\psi(\theta_2) - \psi(\theta_1) \leq -|I|(1 + c^2)^{-1} = -\pi/2$, which contradicts Proposition B.2. Here we have again split the interval of integration at possible zeros of g in I (to deal with singular solutions). \square

For the limiting case, $c = 0$, we have the same conclusion as in Corollary B.3 (seen by using (B.9); this is similar to the zero flux case). We remark that i) of Proposition B.1 obviously does not occur if g is strictly positive. For some models both of i) and ii) may occur. Presumable iii) may only occur at a finite numbers of c 's. If iii) does not occur then presumable LAP would follow by the methods of this paper. The case of Corollary B.3 would be “easy”. The quantum channel corresponding to ii) seems to be similar to the one considered in [CHS].

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