A PRIME GEODESIC ANALOGUE OF THE TWIN PRIME CONJECTURE.

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Abstract. We exhibit the analogy between prime geodesics on hyperbolic Riemann surfaces and ordinary primes. We present asymptotic counting results concerning pairs of prime geodesics whose homology difference is fixed, and explain how this may be considered to be analogous to the classical twin prime conjecture.

1. Introduction

Let M be a compact Riemann surface M of genus $g > 1$. It is a fascinating fact that the norms of the prime closed geodesics on M in many respects are analogous to ordinary primes $p \in \mathbb{N}$. One may think of them as being 'pseudo-primes'. A striking instance of this analogy is the prime geodesic theorem which was proved by Huber [3] and Selberg (see [2]):

$$\pi(x) = \#\{\gamma \in \mathscr{P}(M) | N(\gamma) \leq x\} \sim li(x). \tag{1.1}$$

Here $\mathscr{P}(M)$ is the set of prime closed geodesics (a geodesic is prime if it is not an iterate of another geodesic), $li(x) = \int_1^x 1/log(t)dt$, and $N(\gamma)$ is the norm of γ defined by $N(\gamma) = e^{l(\gamma)}$ where $l(\gamma)$ is the geodesic length of γ.

Consider now $\Phi : \mathscr{P}(M) \to H_1(M, \mathbb{Z})$, i.e. the projection to the first homology group with integer coefficients. Fix $\beta \in H_1(M, \mathbb{Z})$ and let $\pi_\beta(x)$ be the number of prime geodesics γ of norm at most x and with $\Phi(\gamma) = \beta$. Phillips and Sarnak [8] (and immediately following them Adachi and Sunada [1]) found an asymptotic expansion for $\pi_\beta(x)$:

$$\pi_\beta(x) \sim (g-1)^g \frac{x}{\log^{g+1} x}\left(1 + \frac{c_1(\beta)}{\log x} + \frac{c_2(\beta)}{\log^2 x} + \cdots\right). \tag{1.2}$$

The way in which $c_i(\beta)$ depends on the specific homology class β remained unexamined in [8]. We notice that the main term does not depend on β.

In certain applications we would like to understand the dependence of the homology class in this asymptotic expansions. One result in this direction is the following due to Sharp [12]: Fix an isomorphism $\psi : H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ There exist a $2g \times 2g$ positive definite symmetric matrix N of determinant 1 such that

$$\pi_\beta(x) = \frac{e^{-\left\langle \psi(\beta), N^{-1}\psi(\beta)\right\rangle / 2\sigma^2 \log x}}{(2\pi\sigma^2 \log x)^g} li(x) + o\left(\frac{x}{\log^{g+1}(x)}\right), \tag{1.3}$$

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where $\sigma^{-2} = 2\pi(g-1)$ and the implied constant is *independent* of β. The main point is of course the independence of β in the error term, since without this (1.3) reduces to a statement about the main term in a asymptotic expansion a la (1.2).

On average we can get much better error terms. To explain what is known we set up some terminology: Let $\|r\|_m = \max\{|r_i|\}$ be the max norm on $\mathbb{R}^{2g}$. We say that a subset $B \subseteq H_1(M,\mathbb{Z})$ has asymptotic density $d(B)$ if the limit

$$\lim_{x\to\infty} \frac{\#\{\beta \in B | \, \|\psi(\beta)\|_m \leq x\}}{\#\{\beta \in H_1(M,\mathbb{Z}) | \, \|\psi(\beta)y\|_m \leq x\}}$$

exists and equals $d(B)$, i.e. if the image in $\mathbb{Z}^{2g}$ under ψ has asymptotic density in $\mathbb{Z}^{2g}$. Well known examples of subsets that has asymptotic density are

- $B_1 = \{\beta_1, \ldots, \beta_m\}$
- $B_2 = \{\beta \in H_1(M,\mathbb{Z}) | \psi(\beta)_i \equiv a_i \mod l_i\}$
- $B_3 = \{\beta \in H_1(M,\mathbb{Z}) | gcd(\psi(\beta)_1, \ldots \psi(\beta)_{2g}\}$
- $B_4 = \psi^{-1}(A)$ where A is a random set in $\mathbb{Z}^{2g}$,

where the finite set B_1 has density 0, the set B_2, which is just a shifted sub-lattice, has density $(l_1 \cdots l_{2g})^{-1}$, the set B_3 has density $\zeta(2g)^{-1}$, and the random set B_4 has density 1/2.

Petridis and Risager [6] proved that for $B = B_i$, $i = 1, \ldots, 4$

$$\sum_{\substack{\beta \in B \\ \|\psi(\beta)\|_m \leq c \log x}} \left(\pi_\beta(x) - \frac{e^{-\left\langle \psi(\beta), N^{-1}\psi(\beta)\right\rangle/2\sigma^2 \log x}}{(2\pi\sigma^2 \log x)^g} li(x) \right) = o(li(x)) \tag{1.4}$$

where c is a constant depending only on M. Hence on average over such sets the error term in (1.3) is of order $x/\log^{2g+1}(x)$. The range in the sum is essentially best possible: For β with $\|\psi(\beta)\|_m > c \log x$ we have $\pi_\beta(x) = 0$. From (1.4) follows an equidistribution result concerning geodesics in (large) sets of homology classes: When $B = B_i$, $i = 1, \ldots, 4$

$$\frac{\pi_B(x)}{\pi(x)} \to d(B) \text{ as } x \to \infty. \tag{1.5}$$

Here $\pi_B(x)$ is the number of prime closed geodesics with norm at most x and homology class $\Phi(\gamma) \in B$.

One may investigate what happens if we consider pairs – or more generally k-tuples – of prime closed geodesics. Pollicott and Sharp [10] did so in the following way: Let $a_1, \ldots, a_g, b_1, \ldots b_g$ be a fundamental set of generators for the fundamental group $\pi_1(M)$ (see section 3) The conjugacy classes of $\pi_1(M)$ are in one to one correspondence with closed geodesics on M. For a closed geodesic we let $|\gamma| = \min\{\mathrm{wl}\,(g)\, g \in \{\gamma\}\}$ where $\{\gamma\}$ is the conjugacy class associated with the closed geodesic γ and $\mathrm{wl}\,(\gamma)$ is the word length of g in the fundamental set of generators. Pollicott and Sharp used sub-shifts of finite type and the thermodynamic formalism to prove the following pair correlation result: there exist a constant c such that for any $a < b$

$$\#\{(\gamma, \gamma') | \, |\gamma|, |\gamma'| \leq n, \, a \leq l(\gamma) - l(\gamma') \leq b\} \sim c(b-a)\frac{e^{2n}}{n^{5/2}}, \tag{1.6}$$

in the limit $n \to \infty$. We notice that in terms of the 'pseudo-primes' $N(\gamma)$ Pollicott and Sharp are looking at *quotients of norms* in finite intervals. For a somewhat different type of results concerning pairs see [9].

In this paper we study a more geometric counting functions for pairs of geodesics. More precisely we consider the counting function for pairs of prime closed geodesics with norm at most x and fixed homology difference:

$$\pi_2^\beta(x) = \#\{\gamma_1, \gamma_2 \in \mathscr{P}(M) | N(\gamma_i) \leq x,\, \Phi(\gamma_2) - \Phi(\gamma_1) = \beta\}. \tag{1.7}$$

This counting function is geormetric in the sense that the ordering of elements is according to the geodesic length. We will prove the following result:

Theorem 1.1. *Let $\beta \in H_1(M, \mathbb{Z})$.*

$$\pi_2^\beta(x) \sim \frac{(g-1)^g}{2^g} \frac{x^2}{\log^{g+2}(x)}$$

in the limit $x \to \infty$.

In particular there are infinitely many pairs of prime geodesics with fixed homology difference. One may think of this as a hyperbolic Riemann surface version of the twin prime conjecture. For further explanation as to this analogy we refer to section 2. We can now ask how the error term depends on the specific homology class β. We show the following result:

Theorem 1.2. *Let $\beta \in H_1(M, \mathbb{Z})$.*

$$\pi_2^\beta(x) = \frac{1}{2^g} \frac{e^{-\left\langle \psi(\beta), N^{-1}\psi(\beta)\right\rangle / 2\sigma^2 \log(x)}}{(2\pi\sigma^2 \log(x))^g} \frac{x^2}{\log^2(x)} + o\left(\frac{x^2}{\log^{g+2}(x)}\right)$$

when $x > 3$, where the implied constant is independent of β with $\|\psi(\beta)\|_m = o(\sqrt{\log x})$.

As with Sharps result (1.3) the main point in Theorem 1.2 is the existence of an error term which is independent of β. Theorem 1.1 follows trivially from Theorem 1.2.

Remark 1.3. The geometry of the surface M is intimately linked with the spectrum of the Laplacian of the surface considered as a Riemannian manifold. This link is evident from the Selberg trace formulae which relates the lengths of closed geodesics with the eigenvalues of the Laplacian in a summation formulae. (See (3.7) below). This 'duality' between the length spectrum and the Laplace spectrum has proven itself extremely useful both in the study of eigenvalues (e.g. Weyl's law (see e.g. [13, §4.4])) as well as in the study of the lengths of geodesics which is what we investigate in the present work. We use the Selberg trace formulae to count primes in a homology class (a technique developed by Phillips and Sarnak [8]), and we keep track of the dependence on the specific homology class in the error terms. We then analyze how these error terms contribute to the relevant sum. A central point is to see cancellation in an exponential sum arriving from these error terms. Such a cancellation was also a key ingredient in [6].

Remark 1.4. The fact that we are considering surfaces of fixed negative sectional curvature -1, is not essential. If M has variable negative curvature we can combine the ideas of this paper with the ideas developed by Sharp [12], to get results similar to theorems 1.1 and 1.2. In this case the proof uses the thermodynamic formalism instead of the Selberg trace formulae. It is also possible to obtain similar results for free groups using ideas by Petridis and Risager [6, 7]. A crucial ingredient in all these generalizations is to see cancellation in a certain exponential sum.

Remark 1.5. The techniques used in this paper may be used to study counting results of a more general type than the ones we consider. Consider for instance $A \subseteq H_1(M,\mathbb{Z})^k$. We may then consider the counting function

$$\#\{(\gamma_i) \in \mathscr{P}(M)^k | N(\gamma_i) \leq x,\, (\phi(\gamma_i)) \in A\} \tag{1.8}$$

This may be rewritten as

$$\sum_{(\alpha_i)_{i=1}^k \in A} \prod_{i=1}^k \pi_{\alpha_i}(x)$$

By using good expansions for $\pi_{\alpha_i}(x)$ it is now possible to study (1.8). To prove Theorem 1.2 we develop and analyze this in full for

$$A = \{(\alpha_1, \alpha_2) \in H_1(M,\mathbb{Z})^2 | \alpha_2 - \alpha_1 = \beta\}$$

The techniques certainly apply to much more general sets. We hope this paper will be facilitating for anyone interested in such questions.

The paper is organized as follows: In section 2 we describe how the results in this introduction may be seen as analogues of statements in analytic number theory. Section 3 briefly describe the technique developed by Phillips and Sarnak (combined by an idea of Sharp [12]) to count primes in a specific homology class. In the following section we describe how this may be transformed into counting prime pairs with fixed homology difference. In Section 5 we find the main and error term of a counting function with certain weights, and in Section 6 we explain how to use multi-variable summation by parts to remove these weights. Then follows an appendix with some elementary estimates on multidimensional Riemann sums that we have not been able to find a good reference for.

2. Ordinary primes in arithmetic progressions

There is nothing new in this section. Its purpose is to emphasize how (almost) all the results mentioned in the introduction are analogues of classical results or conjectures in analytic number theory. Readers not interested in such connections should feel free to move to the next section, as the rest of the paper does not depend directly on this section. We quote from [4] but most of the results can be found in any solid textbooks on analytic number theory.

Let $\Pi(x) = \#\{p \leq x\}$ be the number of primes less than or equal to x. The prime number theorem [4, Section 2.1] proved by Hadamard and de la Vallée Poussin asserts that

$$\Pi(x) \sim li(x). \tag{2.1}$$

The theorem of Huber and Selberg (1.1) is analogous to (2.1).

Given a primitive conjugacy class a mod q i.e. $(a,q) = 1$ we let $\Pi(x; a, q)$ be the number of primes less than x with $p \equiv a$ mod q. The main result about primes in arithmetic progressions is ([4, (17.2)])

$$\Pi(x; q, a) \sim \frac{li(x)}{\Phi(q)}, \tag{2.2}$$

where $\Phi(q) = \#\{1 \leq a < q | (a,q) = 1\}$ is the Euler totient. The result of Phillips and Sarnak (1.2) is analogous to (2.2).

In applications to other problems involving primes it is of great interest to know how the error term in (2.2) depends on q and x. A first result in this direction is the Siegel-Walfisz theorem [4, Corollary 5.29] which states that for all $A > 0$ $a, q \in \mathbb{N}$, $(a, q) = 1$

$$\Pi(x; q, a) = \frac{li(x)}{\Psi(q)} + O\left(\frac{x}{log^A(x)}\right), \tag{2.3}$$

when $x > 2$ where the implied constant *depends only* on A. We like to think of Sharps theorem (1.3) as analogous to this result.

The extremely potent idea of taking averages over q and a to get better bounds on the error term on average has been used very successfully in the famous theorem of Bombieri and Vinogradov [4, Theorem 17.1]:

Theorem 2.1. *(Bombieri-Vinogradov) For any $A > 0$ there exist $B > 0$ such that*

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \Pi(x; q, a) - \frac{li(x)}{\Phi(q)} \right| = O\left(\frac{x}{\log^A(x)}\right)$$

where $Q = x^{1/2} log^{-B}(x)$. The implied constant depends only on A.

Conjecturally (Elliot-Halberstram) we can take $Q = x^{1-\epsilon}$. In many applications Theorem 2.1 is an excellent substitute for the Generalized Riemann hypothesis which says that $\Pi(x, q, a) = li(x)/\Psi(q) + O(x^{1/2+\epsilon})$ The large range $q \leq x^{1-\epsilon}$ can be handled on average if we allow averages in a also (See [4, Theorem 17.2]):

Theorem 2.2. *(Barbon, Davenport, Halberstram) For any $A > 0$ there exist $B > 0$ such that*

$$\sum_{q \leq Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \left(\Pi(x; q, a) - \frac{li(x)}{\Phi(q)} \right)^2 = O\left(\frac{x}{\log^A(x)}\right)$$

where $Q = xlog^{-B}(x)$. The implied constant depends only on A.

The theorem of Petridis and Risager is analogous to Theorems 2.1 and 2.2.

A folklore conjecture says that there are infinitely many twin primes i.e. primes p such that $p+2$ is a prime. This conjecture was quantified by Hardy and Littlewood who conjectured that

$$\#\{p_1, p_2 \leq x | p_2 - p_1 = 2\} \sim 2c_2 \int_1^x \frac{1}{\log^2(t)} dt \tag{2.4}$$

where $c_2 = \prod_{p>2}(1-(p-1)^{-2})$. We could prove it if we where able to handle certain linear combinations of $\Pi(x; q, a) - li(x)/\Psi(q)$ See ([4, Section 13.1]). Certainly the Montgomery conjecture – $\Pi(x; q, a) = li(x)/\Psi(q) + O(x^{1/2+\epsilon}/q^{1/2})$ – would give it immediately. Unfortunately we are not able to handle the relevant linear combinations and the twin prime conjecture remains completely open.

Theorem 1.1 is analogous to the Conjecture (2.4), and its proof goes along the same lines as what one would like to do for primes. But for prime geodesics the β dependence of $\pi_\beta(x)$ can be understood well enough that we can prove which contributions give error terms and which contribution gives the main term in the relevant linear combination.

3. Counting prime closed geodesics in homology classes

In this section we set up some notation and explain how the Selberg trace formula can be used to cound geodesics in a homology class. We then quote a result from Petridis and Risager [6] derived using this technique, which is the starting point of our current investigation.

Any compact Riemann surface of genus $g > 1$ without boundary may be realized as $M = \Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ is a strictly hyperbolic discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}$ by linear fractional transformations. The surface M has fundamental group $\pi_1(M) = \Gamma$. The closed oriented geodesics (through $m \in M$) are in one to one correspondence with the conjugacy classes of Γ by the following recipe: Pick a base point $z_0 \in \mathbb{H}$ above m. From a conjugacy class $\{\gamma\}$ we project $(\operatorname{mod}\Gamma)$ the geodesic in $\mathbb{H}$ from z_0 to γz_0 to M which gives a closed geodesic on M.

The group Γ has a fundamental set of generators i.e. a set of generators

$$a_1, \dots, a_g, b_1, \dots, b_g \in \Gamma$$

with one defining relation

$$[a_1, b_1] \cdots [a_b, b_g] = 1$$

where $[a, b]$ is the commutator of a and b.

We let C_i, $i = 1, \dots g$, be the geodesic induced by a_i and C_{g+i}, $i = 1, \dots g$, be the geodesic induced by b_i (The fundamental generators lie in different conjugacy classes so $i \neq j$ implies $C_i \neq C_j$). The first homology group $H_1(M, \mathbb{Z})$ is isomorphic to the free group over $\mathbb{Z}$ of $C_1, \dots, C_{2g}$, i.e.

$$H_1(M, \mathbb{Z}) \cong \left\{ \sum m_i C_i | m_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^{2g}$$

The exist a basis for the space of harmonic 1-forms which is dual to $C_1, \dots C_{2g}$ in the sense that

$$\int_{C_i} \omega_j = \delta_{ij}. \tag{3.1}$$

These lift to harmonic differentials $\alpha_i = \Re(f_i(z)dz)$ on $\mathbb{H}$ where $f_i(z)$ is a holomorphic form of weight 2 with respect to Γ. Then $\gamma \in \Gamma$ induces a geodesic with homology $\sum m_i C_i$ if and only if

$$\phi(\gamma) := \left(\int_{z_0}^{\gamma z_0} \alpha_1, \dots, \int_{z_0}^{\gamma z_0} \alpha_{2g} \right) = (m_1, \dots, m_{2g}). \tag{3.2}$$

We notice that $\phi(\gamma)$ does not depend on the choice of path or of the choice of z_0. Consider the unitary characters on Γ defined by

$$\begin{array}{rcccc} \chi_\epsilon & : & \Gamma & \to & S^1 \\ & & \gamma & \mapsto & e^{2\pi i \langle \phi(\gamma), \epsilon \rangle} \end{array}. \tag{3.3}$$

where $\epsilon \in \mathbb{R}^{2g}$ and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

Consider now the set of the set of square-integrable χ_ϵ-automorphic functions, i.e. the set of $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma z) = \chi_\epsilon(\gamma) f(z) \tag{3.4}$$

and

$$\int_F |f(z)|^2 \, d\mu(z) < \infty, \tag{3.5}$$

where F is a fundamental domain for $\Gamma\backslash\mathbb{H}$. Let L_ϵ denote the Laplacian defined as the closure of

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{3.6}$$

defined on smooth compactly supported functions satisfying (3.4) and (3.5). The Laplacian is self-adjoint and its spectrum consists of a countable set of eigenvalues $0 \le \lambda_0(\epsilon) \le \lambda_1(\epsilon) \le \ldots$. We write $\lambda_j(\epsilon) = 1/4 + r_j^2(\epsilon) = s_j(\epsilon)(1-s_j(\epsilon))$. All our geodesic counting results has their origin in the Selberg trace formula for L_ϵ which relates the Laplace spectrum with the length spectrum in a very precise way. (See [11, 2]):

$$\begin{aligned}\sum_j \hat{h}(r_j(\epsilon)) =& 2(g-1)\int_{-\infty}^{\infty} r\tanh(\pi r)\hat{h}(r)dr\\ &+\sum_{\{\gamma\}}\frac{\chi_\epsilon(\gamma)l(\gamma)}{k\sinh(l(\gamma)/2)}h(l(\gamma))\end{aligned} \tag{3.7}$$

where h is a smooth even function on $\mathbb{R}$ of compact support, $\hat{h}$ is its Fourier transform and $l(\gamma)$ is the length of the geodesic induced by $\gamma\in\Gamma$. When $\epsilon = 0$ the contribution from $r_0(0)$ should be counted twice. One of the main ideas in [8] is that by multiplying (3.7) with $\exp(-2\pi i\langle\psi(\beta),\epsilon\rangle)$ and then integrating over the whole character variety (i.e. over $\epsilon\in\mathbb{R}^{2g}\backslash\mathbb{Z}^{2g}$) we pick out exactly those γ on the right hand side of (3.7) with homology class β.

By combining ideas of Sharp [12] and Phillips and Sarnak [8] it is possible to get precise information from (3.7) about

$$R_\beta(x) = \sideset{}{'}\sum_{\substack{N(\gamma)\le x\\ \Phi(\gamma)=\beta}}\frac{l(\gamma)}{\sinh(l(\gamma)/2)} \tag{3.8}$$

(the $'$ on the sum means that we only sum over prime geodesics). Petridis and Risager noticed [6, (2.11)] that up to an error term of decay (independent of β) $x^{-\delta}$

$$\frac{R_\beta(x)}{4\sqrt{x}} - \frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(x)}}{(2\pi\sigma^2\log(x))^g} \tag{3.9}$$

equals

$$\frac{(g-1)^g}{\log^g(x)}\int_{B(\varepsilon\rho\sqrt{\log(x)})}\left(\frac{e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)}}{2s_0(\epsilon/\rho\sqrt{\log(x)})-1}-e^{\langle\epsilon,N\epsilon\rangle}\right)\overline{\chi^\beta_{\epsilon/\rho\sqrt{\log(x)}}}d\epsilon \tag{3.10}$$

for every sufficiently small ε. This will be the starting point for our investigation concerning pairs of prime geodesic.

4. Counting prime pairs with fixed homology difference

In this section we explain how to use the counting technique described in the previous section to count *pairs* of geodesics with restrictions on their homology difference.

We define, for $x_1, x_2 > 1$,

$$\pi_2^\beta(x_1,x_2) := \#\{\gamma_1,\gamma_2\in\mathscr{P}(M)\,|N(\gamma_i)\le x_i, \Phi(\gamma_2)-\Phi(\gamma_2)=\beta\}$$

and we denote $\pi_2^\beta(x) := \pi_2^\beta(x,x)$. We fix $0<k<1$. *We will always assume that*

$$x^k \le x_i \le x. \tag{4.1}$$

An obvious choice is to let $x = \max x_i$. Then (4.1) puts restrictions on $\min x_i$. The restriction (4.1) implies that $\log(x_1)$, $\log(x_2)$, and $\log(x)$ are all of the same size (i.e. $\log(x_1) \asymp \log(x_2) \asymp \log(x)$) The same is true for $\log^{-1}(x_1)$, $\log^{-1}(x_2)$, and $\log^{-1}(x)$. When we, in the following, estimate various sums the error term may depend on k but never on x.

Instead of working with $\pi_2^\beta(x_1,x_2)$ directly it turns out to be more convenient for us to work with something closer related to $R_\beta(x)$. In principle we would like to use

$$\begin{aligned}\pi_2^\beta(x_1,x_2) &= \sum_{\alpha\in H_1(M,\mathbb{Z})} \#\left\{\gamma_1,\gamma_2 \in \mathscr{P}(M) \,\middle|\, \begin{array}{l} N(\gamma_i)\le x_i \\ (\Phi(\gamma_1),\Phi(\gamma_2)) = (\alpha,\beta+\alpha)\end{array}\right\}\\ &= \sum_{\alpha\in H_1(M,\mathbb{Z})} \pi_\alpha(x_1)\pi_{\beta+\alpha}(x_2)\end{aligned}$$

but it turns out to be more convenient to use

$$\begin{aligned}R_2^\beta(x_1,x_2) &:= \sideset{}{'}\sum_{\substack{N(\gamma_i)\le x_i\\ \Phi(\gamma_2)-\Phi(\gamma_1)=\beta}} \frac{l(\gamma_1)l(\gamma_2)}{\sinh\left(l(\gamma_1)/2\right)\sinh\left(l(\gamma_2)/2\right)}\\ &= \sum_{\alpha\in H_1(M,\mathbb{Z})} R_\alpha(x_1)R_{\alpha+\beta}(x_2)\end{aligned} \tag{4.2}$$

The main strategy is now to use (3.9) and (3.10) to find an asymptotic expansion for (4.2) and then use multi-dimensional partial summation to get the expansion for $\pi_2^\beta(x_1,x_2)$.

To be able to handle the infinite sum in (4.2) we start by showing that we only need a finite sum:

Lemma 4.1. *There exist a constant $C>0$ depending only on M such that*

$$\|\phi(\gamma)\|_m \le C l_\gamma \tag{4.3}$$

for all closed geodesics γ (through m).

Proof. This follows directly from [6, Lemma 2.4]. □

From this lemma follows that $R_\beta(x) = 0$ if $\|\psi(\beta)\|_m > C\log(x)$. This implies that in (4.2) we only need to sum over

$$\|\psi(\alpha+\beta)\|_m \le C\log x_2 \tag{4.4}$$

$$\|\psi(\alpha)\|_m \le C\log x_1 \tag{4.5}$$

We note that this also allows us to prove the following result:

Lemma 4.2. $R_\beta^2(x_1,x_2) = 0$ *when* $\|\psi(\beta)\|_m > 2C\log x$.

Proof. Certainly if all summands $R_\alpha(x_1)R_{\alpha+\beta}(x_2)$ vanish the result follows. If $\|\psi(\alpha)\|_m > C\log x_1$ the term $R_\alpha(x_1)$ is zero. If not the triangle inequality gives

$$\begin{aligned}\|\psi(\alpha+\beta)\|_m &\ge \|\psi(\beta)\|_m - \|\psi(\alpha)\|_m\\ &> 2C\log x - C\log x\\ &\ge C\log x_2.\end{aligned}$$

Therefore (4.4) gives the result. □

We may choose a different constant e.g. $C(\beta) = C + \|\beta\|_m$ depending on β such that for $x_2 \geq 3$ we have

$$\{\alpha|\, \|\psi(\alpha+\beta)\|_m \leq C \log x_2\} \subseteq \{\alpha|\, \|\psi(\alpha)\|_m \leq C(\beta) \log x\}$$
$$\{\alpha|\, \|\psi(\alpha)\|_m \leq C \log x_1\} \subseteq \{\alpha|\, \|\psi(\alpha)\|_m \leq C(\beta) \log x\},$$

and we may therefore restrict the sum to $\|\psi(\alpha)\|_m \leq C(\beta) \log x$ i.e. for $x \geq 3$:

$$R_2^\beta(x_1, x_2) = \sum_{\substack{\alpha \in H_1(M,\mathbb{Z}) \\ \|\psi(\alpha)\|_m \leq C(\beta) \log x}} R_\alpha(x_1) R_{\alpha+\beta}(x_2) \tag{4.6}$$

Consider now

$$f_x(\epsilon) = \left(\frac{e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)}}{2s_0(\epsilon/\rho\sqrt{\log(x)}) - 1} - e^{\langle \epsilon, N\epsilon \rangle} \right).$$

We let

$$\begin{aligned} A(\beta, x) = & \quad 4\sqrt{x} \frac{e^{-\langle \psi(\beta), N^{-1}\psi(\beta) \rangle / 2\sigma^2 \log(x)}}{(2\pi\sigma^2 \log(x))^g} \\ B(\beta, x) = & \quad 4\frac{\sqrt{x}}{\log^g(x)} \int_{B(\varepsilon\rho\sqrt{\log(x)})} f_x(\epsilon) \overline{\chi^\beta_{\epsilon/\rho\sqrt{\log(x)}}} d\epsilon. \end{aligned} \tag{4.7}$$

From (3.9) and (3.10) we have

$$R_\beta(x) = A(\beta, x) + B(\beta, x) + O(x^{1/2-\delta}) \tag{4.8}$$

for some $\delta > 0$. The constant δ and the implied constant are absolute. Using this and (4.6) we conclude that when $x \geq 3$

$$\begin{aligned} R_2^\beta(x_1, x_2) = & \sum_{\substack{\alpha \in H_1(M,\mathbb{Z}) \\ \|\psi(\alpha)\|_m \leq C(\beta) \log x}} A(\alpha, x_1) A(\alpha+\beta, x_2) \\ & + \sum_{\substack{\alpha \in H_1(M,\mathbb{Z}) \\ \|\psi(\alpha)\|_m \leq C(\beta) \log x}} A(\alpha, x_1) B(\alpha+\beta, x_2) \\ & + \sum_{\substack{\alpha \in H_1(M,\mathbb{Z}) \\ \|\psi(\alpha)\|_m \leq C(\beta) \log x}} B(\alpha, x_1) A(\alpha+\beta, x_2) \\ & + \sum_{\substack{\alpha \in H_1(M,\mathbb{Z}) \\ \|\psi(\alpha)\|_m \leq C(\beta) \log x}} B(\alpha, x_1) B(\alpha+\beta, x_2) + O_\beta(x^{1-\delta'}) \\ = & \Sigma_1(\beta, x_1, x_2) + \Sigma_2(\beta, x_1, x_2) + \Sigma_3(\beta, x_1, x_2) + \Sigma_4(\beta, x_1, x_2) \\ & \quad + O(C(\beta)^{2g} (x_1 x_2)^{1/2-\delta'}). \end{aligned} \tag{4.9}$$

for some $\delta' > 0$.

5. Finding the main term and error term

We ended the last section by splitting the function $R_2^\beta(x_1,x_2)$ into four different contributions and an error term. In this section we show which contributions are 'big' and which are 'small'. This is the most technical part of our proof of Theorem 1.2. The deepest result seems to be Lemma 5.6 which utilized cancellation in an exponentional sum.

In this section we will prove the following result:

Theorem 5.1. *Let $\beta \in H_1(M,\mathbb{Z})$ and $0<1<k$. Then*

$$R_2^\beta(x_1,x_2) = 16 x_1^{1/2} x_2^{1/2} \frac{e^{-\left\langle \psi(\beta), N^{-1}\psi(\beta)\right\rangle/2\sigma^2 \log x_2}}{(2\pi\sigma^2(\log x_1 + \log x_2))^g} + o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

when $3 < x^k \leq x_i \leq x$, $\|\psi(\beta)\|_m = o(\sqrt{\log x})$ and $x\to\infty$, where the implied constant depends at most on k and M.

Our starting point is the identity (4.9). We start by showing that the main term comes out of Σ_1.

Lemma 5.2.

$$\Sigma_1(\beta,x_1,x_2) = 16 x_1^{1/2} x_2^{1/2} \frac{e^{-\left\langle \psi(\beta), N^{-1}\psi(\beta)\right\rangle/2\sigma^2 \log x_2}}{(2\pi\sigma^2(\log x_1 + \log x_2))^g} + o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

where the implied constant is independent of β, when $\|\psi(\beta)\|_m = o(\sqrt{\log x})$.

Proof. Using (4.9) and (4.7) we easily find that

$$\Sigma_1(\beta,x_1,x_2) = 16\frac{\sqrt{x_1x_2}}{(2\pi\sigma^2)^{2g}}\frac{1}{(\log x_1 \log x_2)^g}\cdot \sum_{\substack{\alpha\in\mathbb{Z}^{2g}\\ \|\alpha\|_m \leq C(\beta)\log x}} e^{-\frac{\left\langle \alpha, N^{-1}\alpha\right\rangle}{2\sigma^2\log x_1}} e^{-\frac{\left\langle (\alpha+\psi(\beta)), N^{-1}(\alpha+\psi(\beta))\right\rangle}{2\sigma^2\log x_2}}$$

We now let $a(x)$ be any function which increases to infinity as x tends to infinity. From Lemma 7.2 (or rather Remark 7.3) we conclude that if we split the sum into contributions from $\|\alpha\|_m \leq a(x)\sqrt{\log x}$ and $a(x)\sqrt{\log x} \leq \|\alpha\|_m \leq C(\beta)\log x$ and bound the last exponential trivially by 1 then the last contribution may go in to the error term in the lemma i.e.

$$\Sigma_1(\beta,x_1,x_2) = 16\frac{\sqrt{x_1x_2}}{(2\pi\sigma^2)^{2g}}\frac{1}{(\log x_1 \log x_2)^g}\cdot \sum_{\substack{\alpha\in\mathbb{Z}^{2g}\\ \|\alpha\|_m \leq a(x)\sqrt{\log x}}} e^{-\frac{\left\langle \alpha, N^{-1}\alpha\right\rangle}{2\sigma^2\log x_1}} e^{-\frac{\left\langle (\alpha+\psi(\beta)), N^{-1}(\alpha+\psi(\beta))\right\rangle}{2\sigma^2\log x_2}} + o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

which we rewrite as

$$= 16\frac{\sqrt{x_1x_2}}{(2\pi\sigma^2)^{2g}}\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log x_2}}{(\log x_1+\log x_2)^g}\cdot$$
$$\sum_{\substack{\alpha\in\mathbb{Z}^{2g}\\ \|\alpha\|_m\le a(x)\sqrt{\log x}}}\frac{e^{-\frac{\left\langle\alpha,N^{-1}\alpha\right\rangle}{2\sigma^2 g(x_1,x_2)}}}{g(x_1,x_2)^g}e^{-2\frac{\left\langle\alpha,N^{-1}\psi(\beta)\right\rangle}{2\sigma^2\log x_2}}$$
$$+\,o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

where

$$g(x_1,x_2)=(\log^{-1}x_1+\log^{-1}x_2)^{-1}=\frac{\log x_1\log x_2}{\log x_1+\log x_2}.$$

We must therefore understand the above sum.

We consider first the case where $\beta=0$. In this case we consider

$$\frac{1}{(2\pi\sigma^2)^g}\sum_{\substack{\alpha\in\mathbb{Z}^{2g}\\ \|\alpha\|_m\le a(x)\sqrt{\log x}}}\frac{e^{-\left\langle\alpha,N^{-1}\alpha\right\rangle/2\sigma^2 g(x_1,x_2)}}{g(x_1,x_2)^g}\tag{5.1}$$

From Lemma 7.2 (or again rather Remark 7.3) we conclude that this converges to 1. It follows that

$$\Sigma_1(0,x_1,x_2)=\frac{16x_1^{1/2}x_2^{1/2}}{(2\pi\sigma^2(\log x_1+\log x_2))^g}+o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

which proves the lemma when $\beta=0$.

The general case follows in the same way if we verify that

$$\sum_{\substack{\alpha\in\mathbb{Z}^{2g}\\ \|\alpha\|_m\le a(x)\sqrt{\log x}}}\frac{e^{-\left\langle\alpha,N^{-1}\alpha\right\rangle/2\sigma^2 g(x_1,x_2)}}{g(x_1,x_2)^g}(1-e^{-2\left\langle\alpha,N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log x_2})=o\left(1\right)\tag{5.2}$$

for some $a(x)$ increasing to infinity and when $\|\beta\|_m=o(\sqrt{\log x})$.

There exist a decreasing function $r(x)$ such that $\|\beta\|_m\le r(x)\sqrt{\log x})$. If we let $a(x)=1/\sqrt{r(x)}$ then there exist a constant C which only depends on N such that

$$(1-e^{-2\left\langle\alpha,N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log x_2})\le C\sqrt{r(x)}\tag{5.3}$$

when $\|\alpha\|_m\le a(x)\sqrt{\log x}$. The bound (5.2) now follows from this and Lemma 7.4. This finishes the proof. □

To be able to bound expressions involving $B(\beta,x)$, i.e. Σ_2,Σ_3, and Σ_4, we need the following lemma:

Lemma 5.3. *Let* $N=\{\langle\omega_i,\omega_j\rangle\}$.

(i) *For every* $\epsilon_0\in\mathbb{R}^{2g}$

$$e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)}\to e^{-\langle\epsilon,N\epsilon\rangle/2}$$

as $x\to\infty$.

(ii) *There exists* $\delta > 0$ *such that for all* $\|\epsilon\| < \delta\rho\sqrt{\log(x)}$.
$$\left| e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)} - e^{-\langle \epsilon, N\epsilon\rangle/2} \right| \leq 2e^{-\langle \epsilon, N\epsilon\rangle/4}.$$

(iii) *There exist constants* $\delta, C > 0$ *such that for all* $\log(x) > 0$, $\|\epsilon\| < \delta \log(x)^{1/32}$,
$$\left| e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)} - e^{-\langle \epsilon, N\epsilon\rangle/2} \right| \leq C\frac{1}{\log(x)^{3/4}}.$$

(iv) *Let* $0 < \nu < 1/4$. *There exist constants* δ_1, δ_2, *such that,*
$$\left| e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)} - e^{-\langle \epsilon, N\epsilon\rangle/2} \right| \leq \frac{e^{-\nu\langle \epsilon, N\epsilon\rangle}}{\log(x)}.$$
when $\log(x) > 0$, $\delta_1\sqrt{\log(\log(x))} < \|\epsilon\| < \delta\sqrt{\log(x)}$.

Proof. Items (i)-(iii) is word by word the proof of [6, Lemma 2.5] with $b = 1/4$ instead of $b = 1/2$. Item (iv) follows from (ii) since
$$e^{-\langle \epsilon, N\epsilon\rangle/4} \leq \frac{e^{-\nu\langle \epsilon, N\epsilon\rangle}}{\log(m)}$$
when $\delta_1\sqrt{\log(\log(x))} < \|\epsilon\|$. □

As a first consequence of Lemma 5.3 (ii) we see that there exist an absolute constant C such that
$$B(\beta, x) \leq C\frac{\sqrt{x}}{\log^g(x)}. \tag{5.4}$$
We also have the obvious
$$A(\beta, x) \leq C\frac{\sqrt{x}}{\log^g(x)}. \tag{5.5}$$

We also get the following lemma:

Lemma 5.4.
$$\int_{B(\varepsilon\rho\sqrt{\log(x)})} |f_x(\epsilon)|\, d\epsilon = o(1)$$

Proof. Since $s_0(\epsilon)$ is even with $s_0(0) = 1$ we have
$$\left|(2s_0(\epsilon) - 1)^{-1} - 1\right| \leq C\,\|\epsilon\|^2 \tag{5.6}$$
when $\|\epsilon\| \leq \varepsilon$. Therefore when $\epsilon \in B(\varepsilon\rho\sqrt{\log(x)})$, $|f_x(\epsilon)|$ is bounded by
$$\left| e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)} - e^{-\langle \epsilon, M\epsilon\rangle/2} \right| + C\left| e^{(s_0(\epsilon/\rho\sqrt{\log(x)})-1)\log(x)} \right| \|\epsilon\|^2 \log(x)^{-1}. \tag{5.7}$$
We can now use Lemma 5.3 (i) (ii) to quote the dominated convergence theorem. The result follows. □

We are now ready to bound the 3 remaining terms of (4.9) which all go into the error term in Theorem 5.1:

Lemma 5.5.
$$\Sigma_i(\beta, x_1, x_2) = o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right), \qquad i = 2, 3$$
where the implied constant is independent of β.

Proof. We see from (4.7) and (4.9) that the function $\Sigma_3(\beta, x_1, x_2)$ equals

$$\frac{16x_1^{1/2}x_2^{1/2}}{\log^g(x_1)}\int\limits_{B(\varepsilon\rho\sqrt{\log(x_1)})} f_{x_1}(\epsilon)\sum_{\substack{\alpha\in H_1(M,\mathbb{Z})\\ \|\psi(\alpha)\|_m\leq C(\beta)\log x_2}}\frac{e^{-\left\langle\psi(\alpha+\beta),N^{-1}\psi(\alpha+\beta)\right\rangle/2\sigma^2\log(x_2)}}{(2\pi\sigma^2\log(x_2))^g}\overline{\chi^{\alpha}_{\epsilon/\rho\sqrt{\log(x_1)}}}d\epsilon.$$

By bounding the character trivially and using Lemma 7.4 we get that the sum is uniformly bounded and we get

$$\Sigma_3(\beta, x_1, x_2) = O\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^g(x_1)}\int_{B(\varepsilon\rho\sqrt{\log(x)})}|f_{x_1}(\epsilon)|\,d\epsilon\right). \tag{5.8}$$

The conclusion now follows from Lemma 5.4 and the fact that

$$\log^{-1}(x_1) \leq k^{-1}\log^{-1}(x_2).$$

The sum $\Sigma_2(\beta, x)$ is handled in the same way. □

The last lemma of this section is maybe the hardest. A pivotal point is the use of cancellation in an exponential sum (See (5.10) below)

Lemma 5.6.

$$\Sigma_4(\beta, x_1, x_2) = o\left(\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)}\right)$$

where the implied constant is independent of β with $\|\psi(\beta)\|_m \leq 2C\log(x)$.

Proof. Using the definitions (4.7) and (4.9) we see that $\Sigma_4(\beta, x_1, x_2)$ equals

(5.9)

$$\frac{16x_1^{1/2}x_2^{1/2}}{\log^g(x_1)\log^g(x_2)}\sum_{\substack{\alpha\in H_1(M,\mathbb{Z})\\ \|\psi(\alpha)\|_m\leq C(\beta)\log x}}\int\limits_{B_1\times B_2} f_{x_1}(\epsilon^1)f_{x_2}(\epsilon^2)\overline{\chi^{\alpha}_{\epsilon^1/\rho\sqrt{\log(x_1)}}\chi^{\alpha+\beta}_{\epsilon^2/\rho\sqrt{\log(x_2)}}}d\epsilon^1 d\epsilon^2$$

where $B_i = B(\varepsilon\rho\sqrt{\log(x_i)})$. We split the integration domain into smaller disjoint pieces. Let $\iota < 1/2$ and $c > 0$:

$$B_1 \times B_2 = D_1 \cup D_2 \cup D_3$$

where

$$\begin{aligned} D_1 &= B(c\log^{\iota}(x_1)) \times B(c\log^{\iota}(x_2)) \\ D_2 &= (B_1\backslash B(c\log^{\iota}(x_1))) \times B_2 \\ D_3 &= B(c\log^{\iota}(x_1)) \times (B_2\backslash B(c\log^{\iota}(x_2)) \end{aligned}$$

The expression (5.9) splits accordingly.

The term from D_2 : The term related to D_2 may be evaluated as follows: We estimate the χ's trivially and get as an upper bound a constant times

$$16x_1^{1/2}x_2^{1/2}(C(\beta)^{2g})\int\limits_{B_1\backslash B(c\log^{\iota}(x_1))}|f_{x_1}(\epsilon)|\,d\epsilon\cdot\int\limits_{B_2}|f_{x_2}(\epsilon)|\,d\epsilon$$

The last integral goes to zero by Lemma 5.4 and the first integral is of exponential decay which is seen as follows: Using (5.7) and Lemma 5.3 (ii) we have $|f_{x_1}(\epsilon)| \leq Ce^{-c(N)\log^{2\iota}x}$ when $\epsilon \in B_1\backslash B(c\log^{\iota}(x_1))$. Here $c(N)$ is some constant depending on N. It follows that the term in (5.9) related to D_2 is $O(C(\beta)^{2g}x_1^{1/2-\nu}x_2^{1/2})$ for some $\nu > 0$. In the range $\|\psi(\beta)\|_m \leq 2C\log(x)$ we have

$$\frac{C(\beta)^{2g}}{x_1^{\nu}} = \frac{(C + \|\psi(\beta)\|_m)^{2g}}{x_1^{\nu}} = o(x_1^{-\nu/2})$$

and the term in (5.9) related to D_2 is $o(x_1^{(1-\nu)/2}x_2^{1/2})$

The term from D_3 : is bounded by the same bound which is proved in the same way.

The term from D_1*:* To handle the sum over D_1 we need to see cancellation in the exponential sum. We have

$$\sum_{\substack{\alpha\in H_1(M,\mathbb{Z})\\ \|\psi(\alpha)\|_m\leq C(\beta)\log x}} \overline{\chi^{\alpha}_{\epsilon^1/\rho\sqrt{\log(x_1)}}\chi^{\alpha+\beta}_{\epsilon^2/\rho\sqrt{\log(x_2)}}}$$
$$= \overline{\chi^{\beta}_{\epsilon^2/\rho\sqrt{\log(x_2)}}} \sum_{\substack{m\in\mathbb{Z}^{2g}\\ |m_i|\leq C(\beta)\log(x)}} e^{2\pi i\left\langle m, \frac{\epsilon^1}{\sqrt{\rho^2\log x_1}} + \frac{\epsilon^2}{\sqrt{\rho^2\log x_2}}\right\rangle}$$

The last sum is

$$\prod_{j=1}^{2g}\sum_{m=-C(\beta)\log x}^{C(\beta)\log x} e^{2\pi i m\left(\frac{\epsilon_j^1}{\sqrt{\rho^2\log x_1}} + \frac{\epsilon_j^2}{\sqrt{\rho^2\log x_2}}\right)}$$

whose norm we may bound in the standard way by

$$\prod_{j=1}^{2g}\frac{1}{e^{2\pi i\frac{\epsilon_j^1}{\sqrt{\rho^2\log x_1}} + \frac{\epsilon_j^2}{\sqrt{\rho^2\log x_2}}} - 1} \leq \frac{1}{4^g\prod_{j=1}^{2g}\left|\frac{\epsilon_j^1}{\sqrt{\rho^2\log x_1}} + \frac{\epsilon_j^2}{\sqrt{\rho^2\log x_2}}\right|}.$$

valid when $\left\|\frac{\epsilon^1}{\sqrt{\rho^2\log x_1}} + \frac{\epsilon^2}{\sqrt{\rho^2\log x_2}}\right\|_m \leq 1/2$. We used the geometric progression and the inequality $|\sin(\pi x)| \geq 2\{x\}$ where $\{x\}$ is the distance between x and the closest integer. Let $k(x_1, x_2) = \sqrt{\log(x_2)/\log(x_1)}$. We notice that

$$k \leq k(x_1, x_2) \leq k^{-1}.$$

We now conclude that

(5.10)
$$\sum_{\substack{\alpha\in H_1(M,\mathbb{Z})\\ \|\psi(\alpha)\|_m\leq C(\beta)\log x}} \overline{\chi^{\alpha}_{\epsilon^1/\rho\sqrt{\log(x)}}\chi^{\alpha+\beta}_{\epsilon^2/\rho\sqrt{\log(x)}}} \leq \frac{\rho^{2g}\log^g(x_1)}{4^g(|\epsilon_1^1 + k(x_1,x_2)\epsilon_1^2|\cdots\left|\epsilon_{2g}^1 + k(x_1,x_2)\epsilon_{2g}^2\right|)}$$

when $\left\|\epsilon^1 + k(x_1,x_2)\epsilon^2\right\|_m \leq \rho\sqrt{\log x_1}/2$.

We see that in the integral (5.9) we need to take special care of the the regions close to $\epsilon_i^1 + k(x_1,x_2)\epsilon_i^2 = 0$ for some i. We do this as follows: Let

$$S(x) = \{(\epsilon_1,\epsilon_2) \in \mathbb{R}^{2g}\times\mathbb{R}^{2g} | \exists i : \left|\epsilon_i^1 + k(x_1,x_2)\epsilon_i^2\right| \leq (C(B)\log^2(x))^{-2g}\}.$$

We then split the contribution from D_1 in (5.9) into contributions according to $D_1 = (D_1\cap S(x)) \cup (D_1 \cap S^c(x))$.

To evaluate the contribution in 5.9) coming from $D_1 \cap S(x)$ we bound the exponential sum trivially and $|f_{x_i}(\epsilon)|$ by a constant (which we can do by Lemma 5.3

(ii)). We therefore get that this contribution is

$$O(x_1^{1/2}x_2^{1/2}C(\beta)^{2g}\mathrm{vol}(S(x)\cap D_1)).$$

Since $\mathrm{vol}(S(x)\cap D_1)=O((C(B)\log^2(x))^{-2g}\log(x)^{(4g-1)\iota})$ the contribution is $o(x_1^{1/2}x_2^{1/2}/log^{2g+\iota}(x))$ which certainly suffices.

To evaluate the contribution coming from $D_1\cap S^c(x)$ we notice that using (5.10) we have to show that

$$\frac{x_1^{1/2}x_2^{1/2}\log^g(x_1)}{\log^g(x_1)\log^g(x_2)}\int\limits_{D_1\cap S^c(x)}\frac{|f_x(\epsilon_1)|\,|f_{x_1}(\epsilon_2)|}{|\epsilon_1^1+k(x_1,x_2)\epsilon_1^2|\cdots\left|\epsilon_{2g}^1+k(x_1,x_2)\epsilon_{2g}^2\right|}d\epsilon^1d\epsilon^2 \tag{5.11}$$

is $o(x_1^{1/2}x_2^{1/2}/\log^{g/2}(x_1)\log^{g/2}(x_2)$ when $\|\psi(\beta)\|_m\le 2c\log(x)$

We split the domain $D_1\cap S^c(x)$ further as the union of

$$\begin{aligned}
L_1(x,x_1,x_2) &:=\{\epsilon\in D_1\cap S^c(x)|\,\|\epsilon_1\|\le v(x_1),\|\epsilon_2\|\le v(x_2)\}\\
L_2(x,x_1,x_2) &:=\{\epsilon\in D_1\cap S^c(x)|\,\|\epsilon_1\|\ge v(x_1),\|\epsilon_2\|\le v(x_2)\}\\
L_3(x,x_1,x_2) &:=\{\epsilon\in D_1\cap S^c(x)|\,\|\epsilon_1\|\le v(x_1),\|\epsilon_2\|\ge v(x_2)\}\\
L_4(x,x_1,x_2) &:=\{\epsilon\in D_1\cap S^c(x)|\,\|\epsilon_1\|\ge v(x_1),\|\epsilon_2\|\ge v(x_2)\}
\end{aligned}$$

where $v(x)=\delta_1\sqrt{log(log(x))}$ with δ_1 as in Lemma 5.3 (iv). We now fix $\iota=1/32$ and $c=\delta$ from Lemma 5.3 (iii). From Lemma 5.3 and (5.7)we find that there exist constants $L>0$ such that

$$|f_{x_1}(\epsilon_1)|\,|f_{x_2}(\epsilon_2)|\le\begin{cases}\frac{L}{\log^{3/4}(x_1)\log^{3/4}(x_2)} & \text{if } (\epsilon^1,\epsilon^2)\in L_1(x,x_1,x_2)\\ \frac{Le^{-\mu\left\|\epsilon^1\right\|^2}}{\log(x_1)\log^{3/4}(x_2)} & \text{if } (\epsilon^1,\epsilon^2)\in L_2(x,x_1,x_2)\\ \frac{Le^{-\mu\left\|\epsilon^2\right\|^2}}{\log^{3/4}(x_1)\log(x_2)} & \text{if } (\epsilon^1,\epsilon^2)\in L_3(x,x_1,x_2)\\ \frac{Le^{-\mu(\left\|\epsilon^1\right\|^2+\left\|\epsilon^2\right\|^2)}}{\log^(x_1)\log(x_2)} & \text{if } (\epsilon^1,\epsilon^2)\in L_4(x,x_1,x_2)\end{cases} \tag{5.12}$$

for some small positive number μ. Let $A_\beta(x)=(C(B)\log^2(x))^{-2g}$. We now use the 4 elementary estimates

$$\begin{aligned}
&\frac{1}{\log^{6/4}(x)}\int\limits_{L_1(x,x_1,x_2)}\frac{1}{\prod_{j=1}^{2g}\left|\epsilon_j^1+k(x_1,x_2)\epsilon_j^2\right|}d\epsilon^1d\epsilon^2\\
&\qquad\le\frac{1}{\log^{6/4}(x)}\prod_{j=1}^{2g}\int\limits_{\substack{|u_i|\le v(x_i)\\ |u_1+k(x_1,x_2)u_2|>A_\beta(x)}}\frac{1}{|u_1+k(x_1,x_2)u_2|}du_1du_2\\
&\qquad=O\left(\frac{log^{2g}(C(\beta)(\log(\log(x)))^{2g})}{\log^{3/2}(x)}\right),
\end{aligned}$$

$$\frac{1}{\log^{7/4}(x)}\int_{L_2(x,x_1,x_2)}\frac{e^{-\mu\left\|\epsilon^1\right\|^2}}{\prod_{j=1}^{2g}\left|\epsilon_j^1+k(x_1,x_2)\epsilon_j^2\right|}d\epsilon^1 d\epsilon^2$$
$$\leq \frac{1}{\log^{7/4}(x)}\prod_{j=1}^{2g}\int_{\substack{|u_1|\leq c\log^{\iota}(x_1)\\ |u_2|\leq v(x_2)\\ |u_1+k(x_1,x_2)u_2|>A_\beta(x)}}\frac{e^{-\mu u_1^2}}{|u_1+k(x_1,x_2)u_2|}du_1du_2$$
$$= O\left(\frac{log^{2g}(C(\beta)(\log(\log(x)))^{2g})}{\log^{7/4}(x)}\right),$$

$$\frac{1}{\log^{7/4}(x)}\int_{L_3(x,x_1,x_2)}\frac{e^{-\mu\left\|\epsilon^1\right\|^2}}{\prod_{j=1}^{2g}\left|\epsilon_j^1+k(x_1,x_2)\epsilon_j^2\right|}d\epsilon^1 d\epsilon^2$$
$$\leq \frac{1}{\log^{7/4}(x)}\prod_{j=1}^{2g}\int_{\substack{|u_2|\leq c\log^{\iota}(x_2)\\ |u_1|\leq v(x_1)\\ |u_1+k(x_1,x_2)u_2|>A_\beta(x)}}\frac{e^{-\mu u_1^2}}{|u_1+k(x_1,x_2)u_2|}du_1du_2$$
$$= O\left(\frac{log^{2g}(C(\beta)(\log(\log(x)))^{2g})}{\log^{7/4}(x)}\right),$$

$$\frac{1}{\log^{2}(x)}\int_{L_4(x,x_1,x_2)}\frac{e^{-\mu(\left\|\epsilon^1\right\|^2+\left\|\epsilon^2\right\|^2)}}{\prod_{j=1}^{2g}\left|\epsilon_j^1+k(x_1,x_2)\epsilon_j^2\right|}d\epsilon^1 d\epsilon^2$$
$$\leq \frac{1}{\log^{6/4}(x)}\prod_{j=1}^{2g}\int_{\substack{|u_i|\leq c\log^{i}(x)\\ |u_1+k(x_1,x_2)u_2|\leq A_\beta(x)}}\frac{e^{-\mu(u_1^2+u_2^2)}}{|u_1+k(x_1,x_2)u_2|}du_1du_2$$
$$= O\left(\frac{log^{2g}(C(\beta)(\log(\log(x)))^{2g})}{\log^{2}(x)}\right).$$

Since $C(\beta)=C+\left\|\psi(\beta)\right\|_m$ it follows from the assumption $\left\|\psi(\beta)\right\|_m\leq 2C\log(x)$ that these 4 expressions are all $o(1)$. This concludes the proof of lemma 5.6. □

Theorem 5.1 now follows from (4.9), Lemmata 4.2, 5.2, 5.5, and 5.6.

6. Using partial summation

In this section we show how to use multi-dimensional partial summation to conclude from Theorem 5.1 our main result:

Theorem 6.1. *Let $\beta\in H_1(M,\mathbb{Z})$ and $0<k<1$.*

$$\pi_2^\beta(x_1,x_2)=\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(x_2)}}{(2\pi\sigma^2(\log x_1+\log(x_2))^g}\frac{x_1x_2}{\log(x_1)\log(x_2)}$$
$$+o\left(\frac{x_1x_2}{\log^{g/2+1}(x_1)\log^{g/2+1}(x_2)}\right)$$

when $3 < x^k \leq x_i \leq x$ *and* $x \to \infty$*, where the implied constant depends at most on* k *and* M*.*

We notice that putting $x_1 = x_2$ we obtain Theorem 1.2 and Theorem 1.1

To prove Theorem 6.1 we let

$$P_2^\beta(x_1, x_2) = \sideset{}{'}\sum_{\substack{N(\gamma_i)\leq x_i\\ \Phi(\gamma_2)-\Phi(\gamma_1)=\beta}} \frac{4\log N(\gamma_1)\log N(\gamma_2)}{\sqrt{N(\gamma_1)}\sqrt{N(\gamma_2)}}).$$

We have a trivial bound

$$P_2^\beta(x_1, x_2) = O(x_1^{1/2}x_2^{1/2}) \tag{6.1}$$

which follows directly from (1.1) by ignoring the condition $\Phi(\gamma_2) - \Phi(\gamma_1) = \beta$.

It is not difficult to see that

$$\begin{aligned} &\left|R_2^\beta(x_1,x_2) - P_2^\beta(x_1,x_2)\right| \\ &\leq \sideset{}{'}\sum_{N(\gamma_i)\leq x_i} \frac{2\log(N(\gamma_1))}{\sqrt{N(\gamma_1)}} \frac{\log(N(\gamma_2))}{N(\gamma_2)\sinh(\log(N(\gamma_2)/2))} \\ &\quad - \sideset{}{'}\sum_{N(\gamma_i)\leq x_i} \frac{2\log(N(\gamma_2))}{\sinh(\log(N(\gamma_2)/2))} \frac{\log(N(\gamma_1))}{N(\gamma_1)\sinh(\log(N(\gamma_1)/2))} \\ &= \sideset{}{'}\sum_{N(\gamma_1)\leq x_1} \frac{2\log(N(\gamma_1))}{\sqrt{N(\gamma_1)}} \sideset{}{'}\sum_{N(\gamma_2)\leq x_2} \frac{\log(N(\gamma_2))}{N(\gamma_2)\sinh(\log(N(\gamma_2)/2))} \\ &\quad - \sideset{}{'}\sum_{N(\gamma_2)\leq x_2} \frac{2\log(N(\gamma_2))}{\sinh(\log(N(\gamma_2)/2))} \sideset{}{'}\sum_{N(\gamma_1)\leq x_1} \frac{\log(N(\gamma_1))}{N(\gamma_1)\sinh(\log(N(\gamma_1)/2))} \\ &= O(x_1^{1/2} + x_2^{1/2}) \end{aligned} \tag{6.2}$$

We used (1.1) again in the last estimate. It follows that Theorem 5.1 holds with $R_2^\beta(x_1, x_2)$ replaced by $P_2^\beta(x_1, x_2)$.

Using multi-dimensional partial summation ([5, Theorem 1.6])we find

$$\begin{aligned} \sideset{}{'}\sum_{\substack{N(\gamma_i)\leq x_i\\ \Phi(\gamma_2)-\Phi(\gamma_1)=\beta}} 1 = &\sideset{}{'}\sum_{\substack{N(\gamma_i)\leq x_i\\ \Phi(\gamma_2)-\Phi(\gamma_1)=\beta}} \frac{4\log N(\gamma_1)\log N(\gamma_2)}{\sqrt{N(\gamma_1)}\sqrt{N(\gamma_2)}} \cdot \frac{\sqrt{N(\gamma_1)}\sqrt{N(\gamma_2)}}{4\log N(\gamma_1)\log N(\gamma_2)} \\ &= \frac{\sqrt{x_1}\sqrt{x_2}}{4\log x_1\log x_2} P_2^\beta(x_1,x_2) \\ &\quad - \frac{\sqrt{x_1}}{4\log x_1}\int_1^{x_2} P_2^\beta(x_1,t_2)m(t_2)dt_2 \\ &\quad - \frac{\sqrt{x_2}}{4\log x_2}\int_1^{x_1} P_2^\beta(t_1,x_2)m(t_1)dt_1 \\ &\quad + \frac{1}{4}\int_1^{x_1}\int_1^{x_2} P_2^\beta(t_1,t_2)m(t_1)m(t_2)dt_2dt_1. \end{aligned} \tag{6.3}$$

where

$$m(t) = \frac{d}{dt}\frac{\sqrt{t}}{\log(t)} = \frac{1}{2\sqrt{t}\log(t)} - \frac{1}{\sqrt{t}\log^2(t)}$$

We then find the asymptotics of the three integrals.

Lemma 6.2. *Let $w \in \mathbb{R}$.*

$$\int_1^{x_1} P_2^\beta(t_1,x_2)(\sqrt{t_1}\log^w(t_1))^{-1}dt_1 = \frac{\sqrt{x_1}}{\log^w(x_1)}P_2^\beta(x_1,x_2)+o\left(\frac{x_1x_2^{1/2}}{\log^{g/2+w}(x_1)\log^{g/2}(x_2)}\right)$$

where the implied constant is independent of β.

Proof. We start by noting that it follows from the trivial bound (6.1)

$$\int_1^{\sqrt{x_1}} P_2^\beta(t_1,x_2)(\sqrt{t_1}\log^w(t_1))^{-1}dt_1 = o(x_1^{3/4}x_2^{1/2})$$

(Clearly $P_2^\beta(x_1,x_2)=0$ for x_1 or x_2 close to 1 since lengths of geodesics does not accumulate at zero, so the integral makes sense even though there seems to be a singularity at 1).

Let

$$m_\beta(x_1,x_2) = 16x_1^{1/2}x_2^{1/2}\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log x_2}}{(2\pi\sigma^2(\log x_1+\log x_2))^g}$$

Consider now

$$\begin{aligned}
&\int_{\sqrt{x_1}}^{x_1} P_2^\beta(t_1,x_2)(\sqrt{t_1}\log^w(t_1))^{-1}dt_1 - \frac{\sqrt{x_1}}{\log^w(x_1)}P_2^\beta(x_1,x_2)\\
&\quad= \int_{\sqrt{x_1}}^{x_1}(P_2^\beta(t_1,x_2)-m_\beta(t_1,x_2))(\sqrt{t_1}\log^w(t_1))^{-1}dt_1\\
&\quad\quad+\int_{\sqrt{x_1}}^{x_1} m_\beta(t_1,x_2)(\sqrt{t_1}\log^w(t_1))^{-1}dt_1 - \frac{\sqrt{x_1}}{\log^w(x_1)}P_2^\beta(x_1,x_2) \qquad (6.4)\\
&\quad= \int_{\sqrt{x_1}}^{x_1}(P_2^\beta(t_1,x_2)-m_\beta(t_1,x_2))(\sqrt{t_1}\log^w(t_1))^{-1}dt_1\\
&\quad\quad+\frac{\sqrt{x_1}}{\log^w(x_1)}m_\beta(x_1,x_2) - \frac{\sqrt{x_1}}{\log^w(x_1)}P_2^\beta(x_1,x_2) + O\left(\frac{x_1x_2^{1/2}}{\log^{g+w+1}(x_1)}\right)
\end{aligned}$$

From Theorem 5.1 and (6.2) follows that there exist a function $g_k(x)$ depending on k', independent of β, and decreasing to zero as $x\to\infty$ such that if $x^{k'}\leq x_i\leq x$ then

$$\left|P_2^\beta(x_1,x_2)-m_\beta(x_1,x_2)\right| \leq g_{k'}(x)\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)} \qquad (6.5)$$

We let $k'=k/2$. Then since we are assuming $x^k\leq x_i\leq x$ we have $x^{k'}\leq t_1,x_2\leq x$ when $t_1\geq\sqrt{x_1}$

If we take absolute values in (6.4) we therefore find

$$\begin{aligned}
|\cdot| &\leq g_{k'}(\sqrt{x})x_2^{1/2}\log^{-g/2}(x_2)\int_{\sqrt{x_1}}^{x_1}\log^{-g/2-w}(t_1))dt_1\\
&\quad+\frac{\sqrt{x_1}}{\log^w(x_1)}g_{k'}(x)\frac{x_1^{1/2}x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)} + O\left(\frac{x_1x_2^{1/2}}{\log^{g+w+1}(x_1)}\right)\\
&= o\left(\frac{x_1x_2^{1/2}}{\log^{g/2+w}(x_1)\log^{g/2}(x_2)}\right)
\end{aligned}$$

□

Lemma 6.3. *Let $w \in \mathbb{R}$.*

$$\int_1^{x_2} P_2^\beta(x_1,t_2)(\sqrt{t_2}\log^w(t_2))^{-1}dt_2 = \frac{\sqrt{x_2}}{\log^w(x_2)}P_2^\beta(x_1,x_2)+o\left(\frac{x_1^{1/2}x_2}{\log^{g/2}(x_1)\log^{g/2+w}(x_2)}\right)$$

where the implied constant is independent of β.

Proof. We claim that

$$\int_{\sqrt{x_2}}^{x_2} m_\beta(x_1,t_2)(\sqrt{t_2}\log^w(t_2))^{-1}dt = m_\beta(x_1,x_2)\frac{\sqrt{x_2}}{\log^w(x_2)}+O\left(\frac{x_1^{1/2}x_2}{\log^{g/2}(x_1)\log^{g/2+w+1}}\right)$$

where the implied constant depends at most on k and w. Using partial integration we see that

$$\begin{aligned}\int_{\sqrt{x_2}}^{x_2}\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(t_2)}}{\log^{g+w}(t_2)}dt_2 &= x_2\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(x_2)}}{\log^{g+w}(x_2)}\\ &- \sqrt{x_2}\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(\sqrt{x_2})}}{\log^{g+w}(\sqrt{x_2})}\\ &- \int_{\sqrt{x_2}}^{x_2}\frac{e^{-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(t_2)}(-\left\langle\psi(\beta),N^{-1}\psi(\beta)\right\rangle/2\sigma^2\log(t_2)-(g+w))}{\log^{g+w+1}(t_2)}dt_2\end{aligned} \tag{6.6}$$

Since $e^{-av}v$ is bounded for $v\in\mathbb{R}_+$ the enumerator of the integrand is bounded (depending on g an w) and the claim follows easily.

Using the claim the proof follows the proof of Lemma 6.2 almost verbatum. □

Lemma 6.4. *Let $w_1, w_2 \in \mathbb{R}$.*

$$\begin{aligned}\int_1^{x_1}\int_1^{x_2} P_2^\beta(t_1,t_2)(\sqrt{t_1}\log^{w_1}(t_1)\sqrt{t_2}\log^{w_2}(t_2))^{-1}dt_1dt_2 \\ = \frac{x_1^{1/2}x_2^{1/2}}{\log^{w_1}(x_1)\log^{w_2}(x_2)}P_2^\beta(x_1,x_2) \\ + o\left(\frac{x_1x_2}{\log^{g/2+w_1}(x_1)\log^{g/2+w_2}(x_2)}\right)\end{aligned}$$

Proof. If we bound $P_2^\beta(x_1,x_2)$ trivially 6.1 we see that we only need to bound the integral over $(t_1,t_2)\in[\sqrt{x_1},x_1]\times[\sqrt{x_2},x_2]$.

$$\begin{aligned}&\left|\int_{\sqrt{x_1}}^{x_1}\int_{\sqrt{x_2}}^{x_2}\cdots dt_1dt_2 - \frac{x_1^{1/2}x_2^{1/2}}{\log^{w_1}(x_1)\log^{w_2}(x_2)}P_2^\beta(x_1,x_2)\right|\\ &\le\left|\int_{\sqrt{x_1}}^{x_1}\int_{\sqrt{x_2}}^{x_2}(P_2^\beta(t_1,t_2)-m_\beta(t_1,t_2))(\sqrt{t_1}\log^{w_1}(t_1)\sqrt{t_2}\log^{w_2}(t_2))^{-1}dt_1dt_2\right|\\ &+\left|\int_{\sqrt{x_1}}^{x_1}\int_{\sqrt{x_2}}^{x_2}\frac{m_\beta(t_1,t_2)}{\sqrt{t_1}\log^{w_1}(t_1)\sqrt{t_2}\log^{w_2}(t_2)}dt_1dt_2 - \frac{x_1^{1/2}x_2^{1/2}}{\log^{w_1}(x_1)\log^{w_2}(x_2)}P_2^\beta(x_1,x_2)\right|\end{aligned}$$

We use (6.6) and a calculation on the last integral:

$$= \left| \int_{\sqrt{x_1}}^{x_1} \int_{\sqrt{x_2}}^{x_2} (P_2^\beta(t_1,t_2) - m_\beta(t_1,t_2))(\sqrt{t_1}\log^{w_1}(t_1)\sqrt{t_2}\log^{w_2}(t_2))^{-1} dt_1 dt_2 \right|$$
$$+ \left| \frac{x_1 x_2}{\log^{w_1}(x_1)\log^{w_2}(x_2)} (m_\beta(x_1,x_2) - P_2^\beta(x_1,x_2)) \right| + O\left(\frac{x_1 x_1}{\log^{g+w_1+w_2}(x)} \right)$$

We then use (6.5)

$$= g_{k'}(x) \int_{\sqrt{x_1}}^{x_1} \int_{\sqrt{x_2}}^{x_2} \frac{t_1^{1/2} t_2^{1/2}}{\log^{g/2}(t_1)\log^{g/2}(t_2)} (\sqrt{t_1}\log^{w_1}(t_1)\sqrt{t_2}\log^{w_2}(t_2))^{-1} dt_1 dt_2$$
$$+ g_{k'}(x) \frac{x_1^{1/2} x_2^{1/2}}{\log^{w_1}(x_1)\log^{w_2}(x_2)} \frac{x_1^{1/2} x_2^{1/2}}{\log^{g/2}(x_1)\log^{g/2}(x_2)} + O\left(\frac{x_1 x_1}{\log^{g+w_1+w_2}(x)} \right)$$
$$= o\left(\frac{x_1 x_2}{\log^{g/2+w_1}(x_1)\log^{g/2+w_2}(x_2)} \right).$$

which finishes the proof the the lemma. □

We are now ready to finish the proof of Theorem 6.1. From (6.3) and lemmata 6.2, 6.3 and 6.4 we find that

$$\begin{aligned} \pi_2^\beta(x_1,x_2) = \frac{1}{16} \frac{x_1^{1/2} x_2^{1/2}}{\log x_1 \log x_2} P_2^\beta(x_1,x_2) \\ + O\left(\frac{x_1^{1/2} x_2^{1/2}}{\log x_1 \log x_2 \log(x)} P_2^\beta(x_1,x_2) \right) \\ + o\left(\frac{x_1 x_2}{\log^{g/2+1}(x_1)\log^{g/2+1}(x_2)} \right) \end{aligned} \tag{6.7}$$

From (6.3) and Theorem 5.1 we find easily

$$P_2^\beta(x_1,x_2) = 0\left(\frac{x_1^{1/2} x_2^{1/2}}{\log^g(x)} \right) \tag{6.8}$$

where the implied constant is independent of β. We conclude that

$$\pi_2^\beta(x_1,x_2) = \frac{1}{16} \frac{x_1^{1/2} x_2^{1/2}}{\log(x_1)\log(x_2)} P_2^\beta(x_1,x_2) + o\left(\frac{x_1 x_2}{\log^{g/2+1}(x_1)\ \log^{g/2+1}(x_2)} \right) \tag{6.9}$$

Using (6.9) Theorem 6.1 follows from Theorem 5.1.

7. Appendix

In this appendix we prove a few elementary results about multi-dimensional Riemann sums that we have not been able to find in the litterature.

Lemma 7.1. *Let $f : \mathbb{R}^k \to \mathbb{R}_+$ be a functions such that for every $S = diag(s_1, \dots, s_k)$, $s_i \in \{\pm 1\}$ the function $f(St)$ decreases in each variable in the set $\{t \in \mathbb{R}^k | t_i \geq 0\}$.*

Let $a > 0$ and assume that $\int\limits_{\|t\|_m \le a} f(t)dt$ exist. Then

$$\left| \sum_{\substack{\alpha\in\mathbb{Z}^k \\ \|\alpha\|_m \le a}} f(\alpha) - \int\limits_{\|t\|_m \le a} f(t)dt \right| \le \sum_{\substack{\emptyset\neq A\subseteq\{1,\dots,n\} \\ A \text{ ordered}}} \int\limits_{\substack{\mathbb{R}^{k-|A|} \\ \|u\|_m \le a}} f_A(u)du + (2^{k+1} - 2 - k(k+3)/2)f(0)$$

where f_A equals f with the i'th coordinate put equal to zero for all $i \in A$.

Proof. In the region $\{t \in \mathbb{R}^k | t_i \ge 0\}$ we bound

$$\int_{R(\alpha)} f(t)dt \le f(\alpha) \qquad \text{valid when } \alpha_i \ge 0 \tag{7.1}$$

$$f(\alpha) \le \int_{R(\alpha-\mathbb{1})} f(t)dt \qquad \text{valid when } \alpha_i > 0 \tag{7.2}$$

where $R(\alpha) = \{t | \alpha_i \le t_i \le \alpha_i + 1\}$, and $\mathbb{1} = (1, \dots, 1)$. Summing these contributions we find

$$\int\limits_{\substack{\|t\|_m \le [a]+1 \\ t_i \ge 0}} f(t)dt \le \sum_{\substack{\|\alpha\|_m \le a \\ \alpha_i \ge 0}} f(\alpha) \tag{7.3}$$

$$\sum_{\substack{\|\alpha\|_m \le a \\ \alpha_i > 0}} f(\alpha) \le \int\limits_{\substack{\|t\|_m \le [a] \\ t_i \ge 0}} f(t)dt \tag{7.4}$$

It follows that

$$0 \le \int\limits_{\substack{\|t\|_m \le a \\ t_i \ge 0}} f(t)dt - \sum_{\substack{\|\alpha\|_m \le a \\ \alpha_i > 0}} f(\alpha) \le \sum_{\substack{\|\alpha\|_m \le a \\ \alpha \in A_k \\ \alpha_i \ge 0}} f(\alpha) \tag{7.5}$$

where $A_k = \{\alpha \in \mathbb{Z}^k | \alpha_i = 0 \text{ for some } i\}$. Doing the same for all the 2^k 'generalized quadrants' we find after adding these contributions that

$$0 \le \int\limits_{\|t\|_m \le a} f(t)dt - \sum_{\substack{\|\alpha\|_m \le a \\ \alpha_i \notin A_k}} f(\alpha) \le 2 \sum_{\substack{\|\alpha\|_m \le a \\ \alpha \in A_k}} f(\alpha) + (2^k - 2)f(0)$$

and adding minus the contribution to the sum from A_k we find

$$\left| \int\limits_{\|t\|_m \le a} f(t)dt - \sum_{\|\alpha\|_m \le a} f(\alpha) \right| \le \sum_{\substack{\|\alpha\|_m \le a \\ \alpha \in A_k}} f(\alpha) + (2^k - 2)f(0). \tag{7.6}$$

For $u \in \mathbb{R}^{k-1}$ we define

$$f_n(u) = f(t) \text{ where } t = (u_1, \dots, u_{n-1}, 0, u_n, \dots, u_n)$$

We can now rewrite (7.6) and get

$$\left| \int\limits_{\|t\|_m \le a} f(t)dt - \sum_{\|\alpha\|_m \le a} f(\alpha) \right| \le (2^k - 2 - (k-1))f(0) + \sum_{n=1}^{k} \sum_{\substack{\alpha\in\mathbb{Z}^{k-1} \\ \|\alpha\|_m \le a}} f_n(\alpha)$$

The result follows by induction. □

We note that Lemma 7.1 generalizes [5, Theorem 1.1] in another way than the generalization [5, Theorem 1.7] We use this to prove

Lemma 7.2. *Let N be a symmetric positive definite $k\times k$ real matrix of determinant 1, and let $C>1$. Then*

$$\sum_{\substack{\beta\in\mathbb{Z}^k\\ \|\beta\|_m\leq Cn}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\left\langle \beta, N^{-1}\beta\right\rangle/2\sigma^2 n} \to 1,$$

as $n\to\infty$.

Proof. Let $a>1$. Consider first

$$\begin{aligned} S_{1,a}(n) = &\sum_{\substack{\beta\in\mathbb{Z}^k\\ \|\beta\|_m\leq a\sqrt{n}}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\left\langle \beta, N^{-1}\beta\right\rangle/2\sigma^2 n}\\ &= \sum_{\substack{\beta\in\mathbb{Z}^k\\ \left\|\beta/\sqrt{n}\right\|_m\leq a}} \frac{1}{(2\pi\sigma^2)^{k/2}} e^{-\left\langle \beta/\sqrt{n}, N^{-1}\beta/\sqrt{n}\right\rangle/2\sigma^2} \frac{1}{n^{k/2}} \end{aligned}$$

This Riemann sum with box volume $n^{-k/2}$ so $S_1(n)$ is convergent to the integral i.e.

$$S_{1,a}(n) \to \frac{1}{(2\pi\sigma^2)^{k/2}} \int_{\|t\|_m\leq a} e^{-\left\langle t, N^{-1}t\right\rangle/2\sigma^2} dt$$

as $n\to\infty$. We notice that since $\det(N)=1$ limit integral converges to 1 as $a\to\infty$.

Considering next

$$S_{2,a}(n) = \sum_{\substack{\beta\in\mathbb{Z}^k\\ a\sqrt{n}\leq\|\beta\|_m\leq Cn}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\left\langle \beta, N^{-1}\beta\right\rangle/2\sigma^2 n}.$$

We use the spectral theorem for real symmetric matrices and the fact that N^{-1} is positive definite to conclude that there exist a constant $\mu>0$ depending only on N such that

$$S_{2,a}(n) \leq \sum_{\substack{\beta\in\mathbb{Z}^k\\ a\sqrt{n}\leq\|\beta\|_m\leq Cn}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\mu\langle \beta,\beta\rangle/2\sigma^2 n}$$

The function

$$f_n(t) = \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\mu\langle t,t\rangle/2\sigma^2 n}$$

satisfies the assumptions of Lemma 7.1 and we conclude that

$$\begin{aligned}
0 \le S_{2,a}(n) \le & \int_{a\sqrt{n}\le \|t\|_m \le Cn} f_n(t)dt \\
& + \sum_{\substack{\emptyset\neq A\subseteq\{1,\dots,k\}\\ A \text{ ordered}}} \int_{\substack{\mathbb{R}^{k-|A|}\\ \|u\|_m\le Cn}} f_{n,A}(u)du + (2^{k+1}-2-k(k+3)/2)f_n(0) \\
\le & \int_{a\sqrt{n}\le \|t\|_m} f_n(t)dt \\
& + 2\sum_{\substack{\emptyset\neq A\subseteq\{1,\dots,k\}\\ A \text{ ordered}}} \int_{\mathbb{R}^{k-|A|}} f_{n,A}(u)du + (2^{k+1}-2-k(k+3)/2)f_n(0)
\end{aligned}$$

We then use substitution of variables to see that there exist a B depending only on N such that this can be bounded as:

$$\begin{aligned}
\le & \int_{a\le \|t\|_m} \frac{1}{(2\pi\sigma^2)^{k/2}} e^{-\mu\langle t,t\rangle/2\sigma^2} dt \\
& + B/\sqrt{n}
\end{aligned}$$

We notice that the integral converges to zero when $a\to\infty$.

Combining the estimates for $S_{1a}(n)$ and $S_{2a}(n)$ we easily get the desired result. □

Remark 7.3. We note that in fact the proof of Lemma 7.2 can easily be changed to prove that for every function $a(x)$ increasing to infinity

$$\sum_{\substack{\beta\in\mathbb{Z}^k\\ \|\beta\|_m\le n^{1/2}a(n)}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\left\langle \beta, N^{-1}\beta\right\rangle/2\sigma^2 n} \to 1$$

as $n\to\infty$, and

$$\sum_{\substack{\beta\in\mathbb{Z}^k\\ \|\beta\|_m\ge n^{1/2}a(n)}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\left\langle \beta, N^{-1}\beta\right\rangle/2\sigma^2 n} \to 0$$

as $n\to\infty$ (i.e. the infinite sum is convergent for every n and the sums converges to zero as n grows to infinity)

We also need a crude upper bound which is uniform in almost all parameters

Lemma 7.4. *Let N be a symmetric positive definite $2g\times 2g$ real matrix of determinant 1. There exist a constant $K>0$ such that for all $A\subseteq\mathbb{Z}^{2g}$, $b\in\mathbb{Z}^{2g}$, $C>1$, $m>1$:*

$$\sum_{\substack{a\in A\\ |a_i|\le Cm}} \frac{1}{(2\pi\sigma^2 m)^g} e^{-\left\langle a+b, N^{-1}(a+b)\right\rangle/2\sigma^2 m} \le K,$$

Proof. The terms are positive so we add a bunch of terms. We enlarge A to $\mathbb{Z}^k$ and if $\|a\|_m\le Cm$ then certainly $\|a+b\|_m\le (C+\|b\|_m)m$ so we have as an upper bound for the sum the new sum

$$\sum_{\substack{a\in\mathbb{Z}^k\\ |a_i|\leq (C+\|b\|_m)n}} \frac{1}{(2\pi\sigma^2 n)^{k/2}} e^{-\langle a, N^{-1}(a)\rangle/2\sigma^2 n}$$

We then proceed as in the proof of Lemma 7.2 and split the sum in $S_{1a}(n) + S_{2a}(n)$. The contribution $S_{1a}(n)$ does not depend on $C + \|b\|_m$ or A and the sum is convergent in n so it is certainly bounded. The final upper bound we gave for $S_{2a}(n)$ does also not depend on $C + \|b\|_m$ or A and it is certainly bounded in n. The conclusion follows. □

References

[1] T. Adachi, T. Sunada, Homology of closed geodesics in a negatively curved manifold. J. Differential Geom. **26** (1987), no. 1, 81–99.

[2] D. Hejhal, The Selberg trace formula for PSL(2, R). Vol. 1. Lecture Notes in Mathematics, 548. Springer-Verlag, Berlin, 1983. vi+516pp.

[3] H. Huber, Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen I, Math. Ann. **138** (1959), 1–26; II Math. Ann. **142** (1960/1961), 385–398; Nachtrag zu II, Math. Ann. **143** (1961), 463—464.

[4] H. Iwaniec E. Kowalski, Analytic number theory, AMS Colloquium Publications vol 53,2004, p. xii+615

[5] E. Krätzel, Lattice points. Mathematics and its Applications (East European Series), 33. Kluwer Academic Publishers Group, Dordrecht, 1988. 320 pp. ISBN: 90-277-2733-3

[6] Y. N. Petridis, M. S. Risager, Equidistribution of geodesics on homology classes and analogues for free groups, Preprint **2005** Submitted

[7] Y. N. Petridis, M. S. Risager, Discrete logarithms in free groups, Proc. AMS, 2006, **134**, 1003-1012.

[8] R. Phillips, P. Sarnak, Geodesics in homology classes. Duke Math. J. 55 (1987), no. 2, 287–297.

[9] N. J.E. Pitt, On pairs of closed geodesics on hyperbolic surfaces, Ann. Inst. Fourier, Grenoble **49**, 1 (1999), 1-25

[10] M. Pollicott, R. Sharp, Correlations for pairs of closed geodesics, Inventiones Mathematicae **163**, 2006, no. 6, p.1-24

[11] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. (N.S.) **20** 1956, 47–87.

[12] R. Sharp, A local limit theorem for closed geodesics and homology. Trans. Amer. Math. Soc. 356 (2004), no. 12, 4897–4908.

[13] A. Venkov, Spectral theory of automorphic functions. A translation of Trudy Mat. Inst. Steklov. **153** (1981). Proc. Steklov Inst. Math. 1982, no. 4 (153), ix+163 pp. 1983.

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