Asymptotic densities of Maass newforms

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Abstract

We define the counting function for Maass newforms of Hecke congruence groups and calculate the three main terms of this counting function. We then give necessary and sufficient conditions for this expansion to have the same shape as if it were counting eigenvalues related to cocompact surfaces. We relate the result to classical instances of the Jacquet-Langlands correspondence.

Key words: Maass newforms, Weyl law, Jacquet-Langlands correspondence.

1 Introduction

For a Fuchsian group of the first kind, Γ, we let Δ_Γ be the automorphic Laplacian related to Γ. The Jacquet-Langlands correspondence (See [12, Chapter 3], [7, Theorem 10.5]) gives (among other things) a correspondence between the λ-eigenspace of Δ_{Γ_c}, and the λ-eigenspace of $\Delta_{\Gamma_{nc}}$. Here Γ_c, Γ_{nc} are certain arithmetical Fuchsian groups of the first kind, where Γ_c is cocompact while Γ_{nc} is non-cocompact but cofinite. This correspondence is usually described using the language of representation theory and adelic trace formulas.

Parts of this theory was reproved in a succession of papers [8], [5], [6] and [18] using classical techniques á la [14]. Here the correspondence is given by an integral transform Θ. The cocompact group Γ_c is the unit group in a maximal order in an indefinite rational quaternion algebra with reduced discriminant D. Hence D is the product of an even number of different primes, and any such number may be realized in this way. Let $\Gamma_{nc} = \Gamma_0(D)$ be the Hecke congruence group of level D.

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Preprint submitted to Elsevier Science *13 November 2003*

In [18] it is proved that Θ gives a bijection between the λ-eigenspace of Δ_{Γ_c} and the λ-new eigenspace of $\Delta_{\Gamma_{nc}}$ when $\lambda \neq 0$. This is proved by a careful comparison of Selberg trace formulas for modular correspondences (Hecke operators) in the two pertinent settings.

For cocompact groups Γ_c the spectral counting function

$$N_{\Gamma_c}(\lambda) = \#\{\lambda_n \leq \lambda | \lambda_n \text{ eigenvalue of } \Delta_{\Gamma_c}\}$$

has an asymptotic expansion of the form

$$N_{\Gamma_c}(\lambda) = \frac{\mathrm{vol}\,(\Gamma_c \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}/\log \lambda), \tag{1}$$

while for congruence groups Γ_{nc}

$$N_{\Gamma_{nc}}(\lambda) = \frac{\mathrm{vol}\,(\Gamma_{nc} \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log \lambda).$$

We notice the difference in the error terms in the compact and the non-cocompact case. Using the 'classical' case of the Jacquet-Langlands correspondence quoted above we find that if we define a spectral counting function $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ which only counts the newforms then when D is the product of an even number of different primes we have

$$N^{\mathrm{new}}_{\Gamma_0(D)}(\lambda) = \frac{\mathrm{vol}\,(\Gamma_c \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}/\log \lambda)$$

i.e. the sort of expansion characteristic to the cocompact case.

We can now ask whether $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ has the same type of expansion for any M not an even number of different primes? When

$$N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda) = c_M\lambda + O(\sqrt{\lambda}/\log \lambda)$$

for some constant c_M we shall say that $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ *is of cocompact type*. By making an asymptotic expansion of the scattering determinant related to $\Gamma_0(M)$ at the halfline $\frac{1}{2} + it$ we can determine asymptotic expansions of $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ with error term $O(\sqrt{\lambda}/\log \lambda)$. We can hence answer the above question. We write $M = t^2 n$ where n is square free.

Theorem A *The spectral counting function $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if and only if at least one of the following holds:*

(1) n contains at least two primes.
(2) n is a prime and $4 \parallel M$.

Hence there are an abundance of cases where $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type but M is not a product of an even number of different primes.

We may therefore try to revert the reasoning and see if $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$ of cocompact type implies that there is a N_{Γ_c} with Γ_c cocompact (or a linear combination of such) that coincides with $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$. In other words: Are there spectral correspondences which are responsible for the remaining cases in Theorem A.

In [16] it is shown that if $\Gamma_c(M)$ is the unit group in an *Eichler order* of level M in an indefinite rational quaternion division algebra with reduced discriminant D then there is a correspondence given by an integral operator such that for $\lambda \neq 0$ a certain λ-new eigenspace of $\Delta_{\Gamma_c(M)}$ is in bijection with the λ-new eigenspace of $\Delta_{\Gamma_0(DM)}$. Hence we still get a bijection with the λ-new eigenspace of $\Delta_{\Gamma_0(DM)}$ if we only 'lift' the λ new eigenforms at the quaternion level.

We choose an approach slightly different from that in [16]. For $D|M'$ we define the λ D-old eigenspace of $\Delta_{\Gamma_0(M')}$ to be the subspace of the λ eigenspace spanned by

$$\left\{ f(dz) \,\middle|\, \begin{matrix} f \text{ in the } \lambda \text{ eigenspace of } \Delta_{\Gamma_0(K)} \\ Kd|M' \quad K \neq M' \quad M'|KD \end{matrix} \right\}.$$

We then define the λ D-new eigenspace to be the orthogonal complement in the λ-eigenspace of $\Delta_{\Gamma_0(M')}$. If $(M, D) = 1$ we can prove, using the trace formula calculations of [16], the following theorem:

Theorem B *When $\lambda \neq 0$ there is a bijection between the λ-eigenspace of $\Delta_{\Gamma_c(M)}$ and the λ D-new eigenspace of $\Delta_{\Gamma_0(MD)}$ given by an integral transform.*

This theorem as well as the one in [16] gives correspondences 'responsible for' the result of Theorem A for DM when $(D, M) = 1$ and D is a product of an even number of different primes. But there are still may cases where $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type that has not been explained in this way. We note however that in all the cases M contains at least two different primes. This leaves some hope that there might be some quaternion division algebra in play. The smallest level that we have not 'explained' is $M = 12$. Are there cocompact groups responsible for the fact that $N^{\text{new}}_{\Gamma_0(12)}(\lambda)$ is of cocompact type or is this purely accidental?

2 Preliminaries

Let Γ be a cofinite discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The group elements acts on the upper half plane, $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ by

$$z \mapsto \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The action is isometric relative to the Poincaré meric $ds^2 = y^{-2}(dx^2 + dy^2)$. The associated measure is $d\mu = y^{-2}dxdy$. The automorphic Laplacian, L_Γ, is the selfadjoint $L^2(\Gamma, d\mu)$ operator defined as the closure of the operator acting on compactly supported functions, f, by

$$L_\Gamma f = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f.$$

If Γ has no cusps then the automorphic Laplacian has infinitely many eigenfunctions and (1) is true.

If Γ has cusps (there can only be finitely many cusps since we assume Γ to be cofinite) the situation is somewhat more complicated. The Roelcke-Selberg conjecture, which claims that also in this case there are infinitely many eigenvalues, seems to have lost credit rather than gained it over the years. From the Selberg trace formula it is possible to derive the following (See [19, Theorem 5.2.1])

$$\begin{aligned} N_\Gamma(\lambda) - \frac{1}{4\pi} \int_{-T}^{T} \frac{\phi_\Gamma'}{\phi_\Gamma} \left(\frac{1}{2} + ir \right) dr \\ = \frac{\operatorname{vol}(\Gamma \setminus \mathbb{H})}{4\pi} \lambda - \frac{k_\Gamma}{\pi} \sqrt{\lambda} \ln \sqrt{\lambda} + \frac{k_\Gamma (1 - \ln 2)}{\pi} \sqrt{\lambda} + O(\sqrt{\lambda} / \ln \sqrt{\lambda}) \end{aligned} \tag{2}$$

where ϕ_Γ is the determinant of the scattering matrix, $\lambda = 1/4 + T^2$ and k_Γ is the number of cusps of Γ. (See [11] for the definition of cusps and the scattering matrix) From this it is clear that in order to estimate the number of eigenvalues it is essential to estimate the logarithmic derivative of the scattering determinant. For congruence subgroups Selberg showed that

$$N_\Gamma(\lambda) = \frac{\operatorname{vol}(\Gamma \setminus \mathbb{H})}{4\pi} \lambda + O(\sqrt{\lambda} \log \lambda). \tag{3}$$

We shall investigate what happens if we restrict ourselves to Hecke congruence groups of level M i.e.

$$\Gamma_0(M) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} c \equiv 0 \mod M \right\}.$$

and only count the eigenvalues corresponding to newforms.

The theory of newforms was originally developed by Atkin & Lehner [3] for holomorphic forms. Their theory can be translated into a similar theory of Maass forms which are the ones we are studying. This has been done independently by various people and details may be found in e.g. [17]. We shall only need one result (Lemma 6 below) and shall hence only sketch enough of the theory for this result to be intelligible.

For any $\lambda > 0$, $M \in \mathbb{N}$ we denote by $A(\lambda, M)$ the λ-eigenspace of $\Delta_{\Gamma_0(M)}$.

Then it is obvious that

$$N_{\Gamma_0(M)}(\lambda) = 1 + \sum_{0<\tilde{\lambda}\leq\lambda} \dim A(\tilde{\lambda}, M), \tag{4}$$

where the sum is certainly finite.

We define the λ-oldspace to be

$$A_{\text{old}}(\lambda, M) := \operatorname{span}\{f(dz) | f \in A(\lambda, K) \quad Kd|M \quad K \neq M\}.$$

This is contained in $A(\lambda, M)$ by the $\mathrm{SL}_2(\mathbb{R})$-invariance of Δ_Γ, and the fact that $f(dz)$ is $\Gamma_0(M)$-invariant when $f(z)$ is $\Gamma_0(K)$-invariant and $Kd|M$. We then define the λ-newspace to be the orthogonal complement in $A(\lambda, M)$ with respect to the inner product

$$(f, g) = \int_{\Gamma_0(M)\backslash\mathbb{H}} f(z)\overline{g(z)}d\mu(z),$$

i.e.

$$A_{\text{new}}(\lambda, M) := A(\lambda, M) \ominus A_{\text{old}}(\lambda, M).$$

We then define new spectral counting functions

$$N^{\text{old}}_{\Gamma_0(M)}(\lambda) := 1 + \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\text{old}}(\tilde{\lambda}, M) \qquad M > 0$$
$$N^{\text{new}}_{\Gamma_0(M)}(\lambda) := \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\text{new}}(\tilde{\lambda}, M) \qquad M > 0.$$

For $M = 1$ we of course define $N^{\text{old}}_{\Gamma_0(1)}(\lambda) = 0$ and $N^{\text{new}}_{\Gamma_0(1)}(\lambda) = N_{\Gamma_0(1)}(\lambda)$.

3 Evaluating the scattering matrix for Hecke congruence groups

As suggested by (2) it is essential to evaluate the logarithmic derivative of the scattering matrix in order to find the asymptotic expansion for the counting function. In this section we estimate the scattering matrix for the congruence groups $\Gamma_0(M)$ by using the following explicit result due to M. Huxley [9]:

Theorem 1 *Let $\phi_M(s)$ be the determinant of the scattering matrix for the Hecke congruence group of level M, $\Gamma_0(M)$, and let Λ_χ be the completed L-*

function of an Dirichlet character mod K ,χ , i.e.

$$\Lambda_\chi(s) = \Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \qquad \textit{when } \Re(s) > 1.$$

Then

$$\phi_M(s) = (-1)^l \left(\frac{A(M)}{\pi^{k_M}}\right)^{1-2s} \prod_{f\in F} \frac{\Lambda_{\overline{\chi_f}}(2-2s)}{\Lambda_{\chi_f}(2s)}$$

where $l \in \mathbb{N}$,

$$F = \left\{ (\chi, m) \,\middle|\, \begin{array}{c} \chi \textit{ primitive Dirichlet character mod } q, n \in \mathbb{N} \\ q|m, mq|M \end{array} \right\}$$

$$A(M) = \prod_{f\in F} \frac{q_f M}{(m_f, M/m_f)}.$$

The number of elements in F equals the number of cusps of $\Gamma_0(N)$ *.*

We now use this to evaluate the integral in (2). We let k_M be the number of cusps in $\Gamma_0(M)$.

Theorem 2 *The counting function* $N_{\Gamma_0(M)}(\lambda)$ *satisfies the following asymptotic formula*

$$\begin{aligned} N_{\Gamma_0(M)}(\lambda) =& \frac{\mathrm{vol}(\Gamma_0(M)\backslash\mathbb{H})}{4\pi}\lambda - \frac{2k_M}{\pi}\sqrt{\lambda}\log\sqrt{\lambda} \\ &+ \frac{1}{\pi}\left[(2-\log 2+\log\pi)k_M - \log(A(M)\right]\sqrt{\lambda} \\ &\qquad + O(\sqrt{\lambda}/\log\sqrt{\lambda}). \end{aligned}$$

In particular we get the following

Corollary 3 *The counting function for* $\Gamma_0(M)$ *is never of cocompact type.*

Proof (Theorem 2). We let $B(M) = \frac{A(M)}{\pi^{k_M}}$. From the above we conclude that

$$\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right) = -2\left(\ln B(M) + \sum_{f\in F}\left(\frac{\Lambda_{\chi_f}'}{\Lambda_{\chi_f}}(1-2it) + \frac{\Lambda_{\chi_f}'}{\Lambda_{\chi_f}}(1+2it)\right)\right).$$

An easy consideration then shows that

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right)dr = \frac{T}{\pi}\ln B(M) + \sum_{f\in F}\frac{1}{\pi}\int_{-T}^{T}\frac{\Lambda_{\chi_f}'}{\Lambda_{\chi_f}}(1+2ir)dr.$$

We must therefore evaluate

$$\int_{-T}^{T} \frac{\Lambda'_{\chi_f}}{\Lambda_{\chi_f}}(1+2ir)dr,$$

and we observe that

$$\int_{-T}^{T} \frac{\Lambda'_{\chi_f}}{\Lambda_{\chi_f}}(1+2ir)d = \frac{1}{2}\int_{-T}^{T} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr + \int_{-T}^{T} \frac{L'_{\chi_f}}{L_{\chi_f}}(1+i2r)\,dr.$$

We shall address each term separately. To evaluate the first term we use Stirling's approximation formula i.e.

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) - \frac{1}{2s} + O(|s|^{-2}),$$

valid for $|\arg(s)-\pi| > \epsilon$ (See [10, A.35]). We see that for $|r| > \epsilon$ we have

$$\begin{aligned}&\left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log|r| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\ &\quad\le \left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log\left|\frac{1}{2}+ir\right| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\ &\quad\quad + \left|\log\left|\frac{1}{2}+ir\right| - \log|r|\right|.\end{aligned}$$

It is easy to check, using Stirling's approximation formula for the first summand, that this is $O(|r|^{-2})$. Hence for any fixed $\epsilon > 0$

$$\begin{aligned}&\frac{1}{2}\int_{-T}^{T} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr \\ &\quad= \frac{1}{2}\int_{\substack{-T \\ |r|>\epsilon}}^{T} \log|r| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\,dr + O\left(\int_{\epsilon}^{T}\frac{dr}{|r|^2}\right) + O(1).\end{aligned}$$

The integral over $(1+i2r)^{-1}$ is $O(\log T)$ and the integral over $i\arg(1/2+ir)$ vanishes. We conclude that

$$\frac{1}{2}\int_{-T}^{T} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr = T\log T - T + O(\log T).$$

To evaluate the integral over the logarithmic derivative of $L_\chi(1+2ir)$ we note that

$$\int_{\epsilon}^{T} \frac{L'_\chi}{L_\chi}(1+2ir)dr = -2i(\log L_\chi(1+2iT)) + C$$

where C is a constant and that the first term is $O(\log T)$ by [1, Theorem 12.24]. We conclude that

$$-\frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'_M}{\phi_M}\left(\frac{1}{2}+ir\right)dr = \frac{T}{\pi}\log B(M) + \frac{k_M}{\pi}(T\log T - T) + O_M(\log(T))$$

which finishes the proof. □

4 Dirichlet convolution

In order to calculate the main terms of $N^{\text{new}}_{\Gamma_0(M)}$ we recall some well known structure theory of arithmetical functions. When $f, g : \mathbb{N} \to \mathbb{C}$ (i.e. arithmetical functions) we define the *Dirichlet convolution*, $f * g : \mathbb{N} \to \mathbb{C}$ to be the arithmetical function

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

We say that f is multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. The structure theory we shall use is the following:

Theorem 4 *The set of arithmetical functions form a commutative group under Dirichlet convolution. The identity element is the function*

$$\begin{aligned} I : \mathbb{N} \to & \quad \mathbb{C} \\ n \mapsto \left[\tfrac{1}{n}\right] = & \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The multiplicative arithmetical functions form a subgroup.

Proof. This follows from [1, Theorems 2.6,2.8,2.14, 2.16]. □

Consider the arithmetical function

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

Then this is a *multiplicative* arithmetical function whose inverse may be calculated to be

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right), \tag{5}$$

where μ is the Möbius function, i.e.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

Another multiplicative arithmetical function we will use is Eulers totient function

$$\Phi(n) = \#\{d \in \mathbb{N} | 1 \le d \le n \wedge (d, n) = 1\}.$$

We can now begin to calculate asymptotic densities of newforms. We cite a result from [17].

Lemma 5

$$\dim A(\lambda, \cdot) = \sigma_0 * \dim A_{\text{new}}(\lambda, \cdot).$$

Proof. This is Theorem 4.6.c) in Chapter III of [17]. $\square$

Let now f_i, $i = 1 \ldots n$ be real positive functions of decreasing order i.e

$$f_{i+1} = o(f_i) \text{ for } i = 1 \ldots n-1.$$

Proposition 6 *Assume that for any* $M \in \mathbb{N}$

$$N_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

Then

$$N^{\text{new}}_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i^{\text{new}}(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

where $c_i^{\text{new}} = c_i * \sigma_0^{-1}$.

Proof. The $M = 1$ case is clear by the definitions of $N^{\text{new}}_{\Gamma_0(1)}(\lambda)$ and $c_i^{\text{new}}(1)$. We observe that by Lemma 5 we have

$$\begin{aligned} N_{\Gamma_0(M)}(\lambda) &= 1 + \sum_{0<\tilde{\lambda}\leq\lambda} \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) \dim A_{\text{new}}(\tilde{\lambda}, K) \\ &= \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) N^{\text{new}}_{\Gamma_0(K)}(\lambda). \end{aligned}$$

By the definition of c_i^{new} we have

$$c_i(M) = \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) c_i^{\text{new}}(K)$$

and therefore

$$\begin{aligned} \left| N^{\text{new}}_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\text{new}}(M) f_i(\lambda) \right| &\leq \left| N_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i(M) f_i(\lambda) \right| \\ &\quad + \sum_{\substack{K|M \\ K\neq M}} \sigma_0\left(\frac{M}{K}\right) \left| N^{\text{new}}_{\Gamma_0(K)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\text{new}}(K) f_i(\lambda) \right|. \end{aligned}$$

Induction in M now gives that this is $\leq Cf_n(\lambda)$ which is the desired result. $\square$

The above proposition together with Theorem 4 enables us to conclude that $c_i^{\text{new}}(N)$ is multiplicative if and only if $c_i(N)$ is multiplicative. It also shows that since we know the expansion of the counting function for eigenvalues of $\Delta_{\Gamma_0(M)}$ for any $M \in \mathbb{N}$ by Theorem 2 it is easy to compute the corresponding counting function for newforms. In the next section we shall do that.

5 The asymptotic expansion of the newform counting function

Theorem 2 now puts us in a situation where Proposition 6 can be applied with

$$\begin{aligned}
f_1(\lambda) &= \lambda \\
f_2(\lambda) &= \sqrt{\lambda}\log\sqrt{\lambda} \\
f_3(\lambda) &= \sqrt{\lambda} \\
f_4(\lambda) &= \sqrt{\lambda}/\log\sqrt{\lambda}.
\end{aligned}$$

From [15, Theorem 1.43] we conclude that

$$k_M = \sum_{d|M} \Phi((d, M/d)) \tag{6}$$

$$\operatorname{vol}(\Gamma_0(M)\backslash\mathbb{H}) = \frac{\pi M}{3} \prod_{\substack{p|M \\ p \text{ prime}}} (1 + p^{-1}). \tag{7}$$

This means that we have explicit expressions for all the terms in Theorem 2 except $A(M)$. We need to know something about the number of primitive Dirichlet characters mod K. We hence define

$$D(K) = \#\{\chi \text{ primitive Dirichlet character mod } K\}.$$

Then we have

Lemma 7 *The arithmetical function $D(K)$ is multiplicative and satisfies*

$$D(K) = (\Phi * \mu)(K).$$

Proof. From [1, Thm. 6.15+Thm. 8.18] we see that $\Phi(K) = \sum_{d|K} D(d) = (u * D)(K)$ where $u(n) = 1$ for $n \in \mathbb{N}$. Since Φ and u are multiplicative we use Theorem 4 to conclude that D is multiplicative. From [1, Thm. 2.1] we see that $u^{-1} = \mu$ so

$$\Phi * \mu = u * D * \mu = u * u^{-1} * D = D$$

which concludes the proof. $\square$

We now calculate c_1^{new}, c_2^{new} and c_3^{new}.

5.1 The first coefficient

We start by calculating $c_1^{\text{new}}(M)$. This is the simplest of the three coefficients.

Proposition 8 *The arithmetical function $v(M) = 12c_1^{\text{new}}(M)$ is multiplicative and satisfies*

$$v(p^n) = \begin{cases} 1 & \textit{if } n = 0 \\ p - 1 & \textit{if } n = 1 \\ p^2 - p - 1 & \textit{if } n = 2 \\ (p^3 - p^2 - p + 1)p^{n-3} & \textit{if } n \geq 3 \end{cases} \tag{8}$$

when p is a prime.

Proof. By using Proposition 6, Theorem 2 and (7) we conclude that

$$M \prod_{\substack{p \mid M \\ p \text{ prime}}} (1 + p^{-1}) = (\sigma_0 * v)(M).$$

Since the left hand side and σ_0 are multiplicative Theorem 4 says that v is multiplicative. By considering the case where $M = p^m$ we see that

$$p^m + p^{m-1} = \sum_{d \mid p^m} \sigma_0(d) v\left(\frac{p^m}{d}\right) = \sum_{i=0}^{m} (i+1) v(p^{m-i}).$$

By applying the theory of generating functions to this relation we find that if

$$f_p(c) = \sum_{n=0}^{\infty} v(p^n) x^n \quad \text{then} \quad f_p(x) = \frac{(1 - x^2)(1 - x)}{1 - px}.$$

By making formal expansion we get (8). $\square$

We note that the fraction of newforms among all automorphic eigenforms on $\Gamma_0(N)$ defined by

$$f(N) = \lim_{\lambda \to \infty} \frac{N^{\text{new}}_{\Gamma_0(N)}(\lambda)}{N_{\Gamma_0(N)}(\lambda)}$$

can be calculated from Proposition 8 to equal

$$f(N) = \frac{v(N)}{N \prod_{p|N}(1+p^{-1})}.$$

From this we can prove that the fraction can be arbitrarily close to 0 and arbitrarily close to 1.

5.2 The second coefficient

We now calculate $c_2^{\text{new}}(M)$. We recall that by Proposition 6 and Theorem 2 we have

$$c_2^{\text{new}}(M) = -\frac{2}{\pi}(k_{(\cdot)} * \sigma_0^{-1})(M)$$

We hence need to have more information about the number of cusps of $\Gamma_0(M)$.

Lemma 9 *The number of cusps, k_M, of $\Gamma_0(M)$ is a multiplicative arithmetical function and satisfies*

$$k_{p^m} = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m = 1 \\ 2p^n & \text{if } m = 2n+1 \text{ where } n > 1 \\ (p+1)p^{n-1} & \text{if } m = 2n \text{ where } n \geq 1 \end{cases} \tag{9}$$

Proof. We noted earlier in (6) that

$$k_M = \sum_{d|M} \Phi((d, M/d)).$$

Let $M_1, M_2 \in \mathbb{N}$ and assume $(M_1, M_2) = 1$. Then

$$\begin{aligned} k_{M_1 M_2} &= \sum_{d|M_1 M_2} \Phi((d, (M_1 M_2)/d)) \\ &= \sum_{d_1|M_1} \sum_{d_2|M_2} \Phi((d_1 d_2, (M_1 M_2)/(d_1 d_2))) \\ &= \sum_{d_1|M_1} \sum_{d_2|M_2} \Phi((d_1, M_1/d_1)(d_2, M_2/d_2)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{d_1|M_1} \Phi((d_1, M_1/d_1)) \sum_{d_2|M_2} \Phi((d_2, M_2/d_2)) \\
&= k_{M_1} k_{M_2}.
\end{aligned}
$$

Hence k_M is multiplicative. The claim about k_{p^m} is clear for $m = 0$ and $m = 1$. Assume $m \geq 2$. We then have

$$
\begin{aligned}
k_{p^m} &= \sum_{i=0}^{m} \Phi((p^i, p^{m-i})) \\
&= \sum_{i=0}^{m} \Phi(p^{\min(i,m-i)}) \\
&= 2 + \sum_{i=1}^{m-1} (p-1) p^{\min(i,m-i)-1}
\end{aligned}
$$

We now assume $m = 2n + 1$.

$$
\begin{aligned}
&= 2 + (p-1) \left(\sum_{i=1}^{n} p^{i-1} + \sum_{i=n+1}^{2n} p^{2n-i} \right) \\
&= 2 + 2(p-1) \sum_{i=0}^{n-1} p^i \\
&= 2 + 2(p-1) \frac{1-p^n}{1-p} = 2p^n.
\end{aligned}
$$

The even case is similar. □

From the above we can now prove the following

Proposition 10 *The arithmetical function ,*$-\frac{\pi}{2} c_2^{\mathrm{new}}(M)$*, is a multiplicative arithmetical function and satisfies*

$$
-\frac{\pi}{2} c_2^{\mathrm{new}}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 2n+1 \\ p-2 & \text{if } m = 2 \\ (p+1)^2 p^{n-1} & \text{if } m = 2n \text{ where } n > 1. \end{cases} \tag{10}
$$

Proof. From Lemma 9 and Theorem 4 follows that $c_2^{\mathrm{new}}(M)$ is multiplicative. From (5) it is easy to see that

$$\sigma_0^{-1}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ -2 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$c_2^{\mathrm{new}}(p^m) = -\frac{2}{\pi}(k_{p^m} - 2k_{p^{m-1}} + k_{p^m}), \text{ when } m \geq 2.$$

Using Lemma 9 it is now easy to check the claim. We omit the details. □

As an easy corollary we get the following

Corollary 11 *The second coefficient, $c_2^{\mathrm{new}}(M)$, is non-zero if and only if $M = t^2$ where $t \in \mathbb{N}$ is not of the form $t = 2t'$ with $(2, t') = 1$.*

5.3 The third coefficient

We finally calculate $c_3^{\mathrm{new}}(M)$. This is the most difficult of the three coefficients.

We start by observing that by Proposition 6 and Theorem 2

$$c_3^{\mathrm{new}}(M) = \frac{1}{\pi}\left((2 - \log 2 + \log \pi)\left(-\frac{\pi}{2}c_2^{\mathrm{new}}(M)\right) - L(M)\right)$$

where

$$L(M) = \left(\log A(\cdot) * \sigma_0^{-1}\right)(M).$$

We hence direct our attention to $L(M)$.

Lemma 12 *Assume $(M_1, M_2) = 1$. Then*

$$L(M_1M_2) = U(M_1)L(M_2) + U(M_2)L(M_1)$$

where

$$U(M) = \sum_{d|M}\sum_{m|d}\sum_{q|(m,\frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M}{d}\right).$$

Proof. We have

$$L(M_1M_2) = \sum_{d|M_1M_2} \log A(d)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right)$$

$$
\begin{aligned}
&= \sum_{d|M_1M_2} \sum_{\substack{q|m \\ mq|d}} D(q) \log\left(\frac{qd}{(m, \frac{d}{m})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m,d/m)} D(q) \log\left(\frac{qd}{(m, \frac{d}{m})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1, \frac{d_1}{m_1})} \sum_{q_2|(m_2, \frac{d_2}{m_2})} \\
&\qquad\qquad D(q_1q_2) \log\left(\frac{q_1q_2d_1d_2}{(m_1m_2, \frac{d_1d_2}{m_1m_2})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d_1d_2}\right).
\end{aligned}
$$

The summand is clearly

$$
D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right) \sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1, \frac{d_1}{m_1})}\right) + \log\left(\frac{q_2d_2}{(m_2, \frac{d_2}{m_2})}\right)\right).
$$

We have

$$
\begin{aligned}
&\sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1, \frac{d_1}{m_1})} \sum_{q_2|(m_2, \frac{d_2}{m_2})} \\
&\qquad\qquad D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1, \frac{d_1}{m_1})}\right)\right) \\
&\quad = U(M_2)L(M_1),
\end{aligned}
$$

from which the identity easily follows. □

It turns out that U is a very nice arithmetical function. In fact we have the following.

Lemma 13 *The function $U(M)$, is a multiplicative arithmetical function and satisfies*

$$
U(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 2n+1 \\ p-2 & \text{if } m = 2 \\ (p^2-2p+1)p^{n-2} & \text{if } m = 2n \text{ where } n > 1. \end{cases} \tag{11}
$$

Proof. Let $M_1, M_2 \in \mathbb{N}$ be coprime. Then

$$
U(M_1M_2) = \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m, \frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right)
$$

$$= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1, \frac{d_1}{m_1})} \sum_{q_2|(m_2, \frac{d_2}{m_2})} D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)$$
$$= U(M_1)U(M_2).$$

Hence U is multiplicative.

Let p be a prime and $m \in \mathbb{N}$. We assume $m \geq 2$ Then

$$\begin{aligned}
U(p^m) &= \sum_{i=0}^{m}\sum_{j=0}^{i}\sum_{l=0}^{\min(j,i-j)} D(p^l)\sigma_0^{-1}\left(p^{m-i}\right) \\
&= \sum_{j=0}^{m-2}\sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=0}^{m-1}\sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=0}^{m}\sum_{l=0}^{\min(j,m-j)} D(p^l)
\end{aligned}$$

Assume $j \leq n-2-j$. Then $j \leq n-1-j \leq n-j$ and we have that all minimum values are j. Hence these terms cancels out. We now assume $m = 2n+1$. Hence we may sum from $j \geq (2n+1)/2 - 1 = n - 1/2$.

$$\begin{aligned}
&= \sum_{j=n}^{m-2}\sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=n}^{m-1}\sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=n}^{m}\sum_{l=0}^{\min(j,m-j)} D(p^l) \\
&= \sum_{j=n}^{m-2}\sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{j=n}^{m-1}\sum_{l=0}^{m-1-j} D(p^l) + \sum_{j=n+1}^{m}\sum_{l=0}^{m-j} D(p^l) + \sum_{l=0}^{m} D(p^l) \\
&= \sum_{l=0}^{m-2-n} D(p^l) - 2\sum_{l=0}^{m-1-n} D(p^l) + \sum_{l=0}^{n} D(p^l) \\
&\quad + \sum_{j=n+1}^{m-2}\left(\sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{l=0}^{m-1-j} D(p^l) + \sum_{l=0}^{m-j} D(p^l)\right) \\
&\quad - 2\sum_{l=0}^{m-1-(m-1)} D(p^l) + \sum_{l=0}^{m-(m-1)} D(p^l) + \sum_{l=0}^{m-m} D(p^l) \\
&= -D(p^n) + \sum_{j=n+1}^{m-2}(-2D(p^{m-1-j}) + D(p^{m-1-j}) + D(p^{m-j})) + D(p) \\
&= -D(p^n) - \sum_{j=1}^{n-1} D(p^j) + \sum_{j=2}^{n} D(p^j) + D(p) \\
&= 0.
\end{aligned}$$

The even case is similar but slightly easier. The $m = 1$ case is also similar. □

By successive use of the two lemmas above we find that

$$L(p_1^{n_1}\dots p_k^{n_k}) = \sum_{i=1}^{k}\left(\prod_{j\in\{1,\dots,k\}\setminus\{i\}} U(p_j^{n_j})\right) L(p_i^{n_i}),$$

when $p_1,\dots,p_k$ are different primes. Notice that $L(p_i^{n_i})$ is of the form $\tilde{m}_i \log p_i$ where $\tilde{m}_i \in \mathbb{Z}$. We also note that $U(M)\in\mathbb{Z}$. Hence L is on the form

$$m_1 \log p_1 + \dots, m_k \log p_k \text{ where } m_i \in \mathbb{Z}.$$

By unique factorization in $\mathbb{N}$ this is zero if and only if $m_i = 0$ for all i's. We would therefore like to know when $L(p^m)$ is zero.

Lemma 14 *The function $L(p^m)$ satisfies*

$$L(p^m) = \begin{cases} 2\left(\sum_{j=0}^{n} D(p^j)\right)\log p & \textit{if } m = 2n+1 \\ \left(\sum_{j=0}^{n-1} D(p^j) + mD(p^n)\right)\log p & \textit{if } m = 2n \\ 0 & \textit{if } m = 0. \end{cases} \tag{12}$$

In particular $L(p^m)$ is never zero, when $m \geq 1$.

Proof. This follows by a lengthy but elementary calculation similar to that in the proof of Lemma 13. □

From the above lemma and the preceding remark we conclude that

$$L(p_1^{n_1}\dots p_k^{n_k}) = 0$$

if and only if $U(p_i^{n_i}) = 0$ for at least two different primes. Since $c_2^{\text{new}}(M) = c_3^{\text{new}}(M) = 0$ if and only if $c_2^{\text{new}}(M) = L(M) = 0$ we have proved the Theorem A which settles the question of when $N_{\Gamma_0(M)}^{\text{new}}(\lambda)$ is of cocompact type.

From Proposition 8 we conclude that

$$N_{\Gamma_0(M)}^{\text{new}}(\lambda) = \frac{1}{12}\lambda + O(\sqrt{\lambda}\log\sqrt{\lambda})$$

if and only if $M \in \{1,2,4\}$. This shows that theorem 2 of [4] cannot be generalized to more general Hecke congruence groups by simply choosing another character.

As mentioned in the introduction particular cases of Theorem 18 follows from the Jacquet-Langlands correspondence. A part of this correspondence is described classically in [18] where the following is proven:

Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra over $\mathbb{Q}$, and let $d = d(\mathcal{O})$ be its (reduced) discriminant. (This is always a square free integer with an even number of prime factors). The norm one unit group $\mathcal{O}^1$ can be viewed as a Fuchsian group *which is cocompact.* Then:

The eigenvalues of the Laplacian on $\mathcal{O}^1 \setminus \mathcal{H}$ *are exactly the same (with multiplicities) as the eigenvalues corresponding to the newspace on* $\Gamma_0(d)\backslash\mathcal{H}$.

Hence if d is the (reduced) discriminant of such a maximal order then $N^{\text{new}}_{\Gamma_0(d)}(\lambda)$ *is* the counting function for the eigenvalues related to a cocompact group, and hence obviously has the corresponding type as predicted by (1). We note that any square free d with an even number of primes may be constructed in this way (See [20, III. Theoreme 3.1]). Our calculation indicates that there might be a similar correspondence in a lot of other cases. The subject of the next section is to describe such correspondences in some cases.

6 D-newforms

We wish to generalize the result in [18]. One way is to proceed as in [16], but we choose a slightly different road. Instead of reducing the domain of definition of the operator in play as in [16] we enlarge the allowed image. To this end we introduce the concept of D-newforms. Assume $D|M$. We define (see also [2]) the (λ, D)-oldspace $A_{D\text{-old}}(\lambda, M)$ to be

$$\text{span}\{f(dz)|f \in A(\lambda, K) \quad Kd|M \quad K \neq M \quad M|KD\}.$$

This is contained in $A(\lambda, M)$ by the $\text{SL}_2(\mathbb{R})$-invariance of Δ_Γ, and the fact that $f(dz)$ is $\Gamma_0(M)$-invariant when $f(z)$ is $\Gamma_0(K)$-invariant and $Kd|M$. As for the usual newform oldform dichotomy we define the (λ, D)-newspace as the orthogonal complement to the (λ, D)-oldspace, i.e.

$$A_{D\text{-new}}(\lambda, M) := A(\lambda, M) \ominus A_{D\text{-old}}(\lambda, M).$$

We note that the space of (λ, M)-oldforms is the usual space of oldforms while the space of $(\lambda, 1)$-oldforms is the empty set. Hence we have

$$\begin{aligned} A_{M\text{-new}}(\lambda, M) &= A_{\text{new}}(\lambda, M) \\ A_{1\text{-new}}(\lambda, M) &= A(\lambda, M) \end{aligned}$$

We also note that $A_{\text{new}}(\lambda, M)$ is a subspace of $A_{D\text{-new}}(\lambda, M)$ We denote the Hecke newforms basis of $A_{\text{new}}(M, \lambda)$ where the elements are normalized to have first coefficient equal to 1 by $f_1^{(M)} \dots f_{m_M}^{(M)}$.

Proposition 15 *Let $\lambda > 0$. If $(D, M/D) = 1$ then $A_{D\text{-new}}(M, \lambda)$ has*

$$B = \{f_i^{(K)}(dz) | \quad dK|M \quad i = 1 \dots m_K \quad D \mid K\}$$

as a basis.

Proof. By [17, Theorem 4.6 c] $A(M, \lambda)$ has as a basis

$$f_i^{(K)}(dz) \quad dK \mid M \quad i = 1, \dots, m_K, \tag{13}$$

so the elements of B are linearly independent. Assume $D \nmid K$. Then since $dK \mid M$ we may make the following factorization $K = K_1 K_2$, $K_1 \mid D$, $K_2 \mid M/D$, $d = d_1 d_2$, $d_1 \mid D$ $d_2 \mid M/D$, where $K_1 d_1 \mid D$ and $K_2 d_2 \mid M/D$. We notice that $K_1 \neq D$ by assumption. By [17, Lemma 4.4 e] we have $f(d_2 z) \in A(\lambda, d_2 K) = A(\lambda, K_1 d_2 K_2) \subset A(\lambda, K_1 M/D)$. If we let $K' = K_1 M/D$ and $d' = d_1$, then $K'd' \mid M$, $M \neq M$ and $M \mid K'D$. Hence by definition $f(d_1(d_2 z)) \in A_{D\text{-old}}(M, \lambda)$. To see that the elements of B is in the orthogonal complement of $A_{D\text{-old}}(M, \lambda)$, we notice that if $f \in A(\lambda, K)$, $Kd|M$, $K \neq M$ and $M \mid KD$ then it may be written in the basis (13) where all the elements have $M/D|K$. But by [17, p.96 l. $9^- - 5^-$] these are all orthogonal to the elements of B. $\square$

We notice that in the above proposition (2) is not true if we have $(D, M/D) > 1$. Consider $M = p^2$ and $D = p$. In this case newforms in $A(\lambda, p)$ are D-oldforms.

It follows that when $(D, M/D)$=1 and $M = M'D$ then

$$\dim A_{D\text{-new}}(M'D, \lambda) = \sum_{K'|M'} \sigma_0 \left(\frac{M'}{K'}\right) \dim A_{\text{new}}(\lambda, DK'), \tag{14}$$

which is a Dirichlet convolution

$$\dim A_{D\text{-new}}(M'D, \lambda) = (\sigma_0 * \dim A_{\text{new}}(\lambda, D \cdot -))(M').$$

Now by (5) we can invert and get

$$\dim A_{\text{new}}(\lambda, DM')) = \sigma_0^{-1} * \dim A_{D\text{-new}}(\lambda, - \cdot D).$$

Now this gives immediately

$$N^{\text{new}}_{\Gamma_0(DM')}(\lambda) = \sigma_0^{-1} * N^{D\text{-new}}_{\Gamma_0(-\cdot D)}(\lambda),$$

where

$$N^{D-\text{new}}_{\Gamma_0(M)}(\lambda) = \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{D\text{-new}}(\lambda, M)$$

is the counting function of D-newforms. We will show in the following section that when $(D, M/D) = 1$ and D is a product of an even number of primes then $N_{\Gamma_0(M)}^{D\text{-new}}$ is not only asymptotically equal to, but in fact identical to a counting function related to a cocompact Fuchsian group of the first kind. This will give an alternative proof of Theorem A when $M = DM'$ where D is the product of an even number of primes and $(D, M') = 1$.

7 A spectral correspondence for Maass waveforms

Let $\mathcal{A}$ be an indefinite rational quaternion division algebra and let D be the discriminant of $\mathcal{A}$. Then D is an even number of different primes [20, III. Theoreme 3.1]. We fix a maximal order $\mathcal{O}$ in $\mathcal{A}$ and fix an isomorphism

$$A_v \simeq \mathrm{M}_2(\mathbb{Q}_v) \qquad \text{for } v \in \{\infty\} \cup \{p | p \text{ prime}, p \nmid D\}, \tag{15}$$

such that we get isomorphims $\mathcal{O}_p \simeq \mathrm{M}_2(\mathbb{Z}_v)$ for all prime $p \nmid D$. Now for each $M \in \mathbb{N}$ with $(D, M) = 1$ we have Eichler orders $\mathcal{O}(M)$ uniquely defined by the following conditions:

(i) $\mathcal{O}(M)_p$ is equivalent to the unique maximal order in $\mathcal{A}_p$ for $p \mid d$.

(ii) $\mathcal{O}(M)_p$ is equivalent to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ M\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ for $p \nmid d$.

The norm 1 unit group $\Gamma_{\mathcal{O}(M)}$ can be identified with a cocompact Fuchsian group through (15) for $v = \infty$ (See [13, Chapter 5] for further details). We consider the λ-eigenspaces $A_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ of $\Delta_{\Gamma_{\mathcal{O}(M)}}$. Let for $(n, MD) = 1$, $\tilde{T}_n : A_{\Gamma_{\mathcal{O}(M)}}(\lambda) \to A_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ be the Hecke operators (See [6, (6.3),(6.4) $(n, MD) = 1$]). These operators form a commuting family of selfadjoint operator and we may choose a basis $f_1 \dots f_k$ of common eigenfuntions. The thetamap $\Theta : \mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda) \to \mathcal{A}_{\Gamma_0(MD)}(\lambda)$, is defined as in [6, (5.1) and (4.8)]. This is a linear integral transformation which, under some assumption about a reference point z_0 has trivial kernel. (See [17, Theorem 1.3 and (6.2)]). This map commutes with the Hecke operators i.e.

$$\Theta \tilde{T}_n f = T_n \Theta f.$$

We have the following fundamental equality

Theorem 16 *Let $\lambda > 0$ and assume $(M, D) = 1$. Then*

$$\mathrm{Tr}\left(T_n|_{A_{D\text{-new}}(DM,\lambda)} \right) = \mathrm{Tr}\left(\tilde{T}_n\Big|_{A_{\Gamma_{\mathcal{O}(M)}}(\lambda)} \right). \tag{16}$$

Proof. From Proposition 15 and [17, p.49 l.1-4] we find

$$\mathrm{Tr}\left(T_n\left|_{A_{D\text{-new}}(DM,\lambda)}\right.\right) = \left(\sigma_0 * \mathrm{Tr}\left(T_n\left|_{A\text{-new}(-\cdot D,\lambda)}\right.\right)\right)(M)$$

From [16] we get that

$$\mathrm{Tr}\left(\tilde{T}_n\Big|_{A_{\Gamma_{\mathcal{O}(M)}}(\lambda)}\right) = \sigma_0 * \mathrm{Tr}\left(\tilde{T}_n\Big|_{A^{\mathrm{new}}_{\Gamma_{\mathcal{O}(\cdot)}}(\lambda)}\right)(M)$$

and

$$\mathrm{Tr}\left(T_n|_{A_{\mathrm{new}}(\lambda,MD)}\right) = \mathrm{Tr}\left(\tilde{T}_n\Big|_{A^{\mathrm{new}}_{\Gamma_{\mathcal{O}(M)}}(\lambda)}\right)$$

The result follows immediately. □

Now we are ready to state the main theorem of this section. We assume that the reference point z_0 which is used in the definition of Θ is chosen such that Θ has trivial kernel.

Theorem 17 *Assume $\lambda > 0$. Then Θ gives a bijection between $\mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ and $\mathcal{A}_{D\text{-new}}(\lambda, MD)$.*

Proof. We start by noticing that Theorem 16 with $n = 1$ gives us that the two spaces have the same dimension. The proof goes as in [17, III 6.] and we shall not repeat the argument in detail. We only need to replace d_B in [17] with DM. Note also that the argument on p. 61 l. 14-20 generalizes simply by noticing that if $f^{(DK')}(z)$ is a newform in $A_{\mathrm{new}}(DK', \lambda)$ with Hecke eigenvalues $\tau(p)$ for p any prime, then in the basis given in [17, Theorem 4.6] only $f_i^{(DK')}(dz)$, $dK'|M$ have the right eigenvalues for $p \nmid DM$ by [17, Theorem 4.6.d) and Lemma 4.4 g)]. Therefore - using the notation from [17] - $\Theta(\tilde{f}_j)$ is in the span of the elements $f_i^{(DK')}(dz)$. Hence Θ maps $\mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ into $\mathcal{A}_{D\text{-new}}(\lambda, MD)$ by Proposition 15, and the result follows since Θ has trivial kernel. □

We note that this is Theorem B Notice that if $M = 1$ this is the result quoted from [17].

Acknowledgements

I am grateful to Professors A. B. Venkov and E. Balslev for inspiring conversation and helpful advise, and to Professor R. Bruggeman for useful comments on an early draft of this paper. I am also grateful to A. Strömbergsson for generously sharing with me unpublished ideas and results.

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