

LEFSCHETZ THEOREMS AND DEPENDENT RATIONAL POINTS ON CURVES OVER FINITE FIELDS.

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ABSTRACT. For a smooth curve C over a finite field \mathbb{F}_q , we prove that the probability that a randomly chosen set of τ rational points impose dependent conditions on a given linear system of dimension τ is asymptotically equal to $\frac{1}{q}$.

The proof involves a geometric construction and a Lefschetz theorem for quasi-projective varieties.

The result has applications in the assessment of the performance of decoding algorithms for algebraic geometry codes.

Let C be a smooth and absolutely irreducible curve of genus g defined over the finite field \mathbb{F}_q and let D be a \mathbb{F}_q -rational divisor on C with $l(D) = \tau$.

Let X be τ -tuples of pairwise different points on C , i.e.

$$X = \{(P_1, \dots, P_\tau) \mid P_i \neq P_j \text{ for } i \neq j\}$$

and let $\Gamma \subset X$ be τ -tuples of pairwise different points on C failing to impose independent conditions on the linear system of divisors equivalent to D . Specifically, if $\overline{\mathbb{F}}_q(C)$ denotes the field of rational functions on C , then

$$\Gamma = \{(P_1, \dots, P_\tau) \in X \mid \exists f \in \overline{\mathbb{F}}_q(C) : \text{div}(f) + D - (P_1 + \dots + P_\tau) \geq 0\}.$$

Let $|X(\mathbb{F}_{q^j})|$ and $|\Gamma(\mathbb{F}_{q^j})|$ denote the number of \mathbb{F}_{q^j} -rational points on X and Γ . Then we prove that

Theorem 1. *In the notation above assume that $\deg(D) \geq 2g + 1$ and let $\tau = \deg(D) + 1 - g$. Assume $\Gamma \neq \emptyset$. There is a constant c (independent of j), such that*

$$\left| |X(\mathbb{F}_{q^j})| - q^j |\Gamma(\mathbb{F}_{q^j})| \right| \leq c (q^j)^{\frac{\tau+1}{2}}. \quad (1)$$

The bounding term $c (q^j)^{\frac{\tau+1}{2}}$ can not in general be replaced by a smaller power of q^j , as the following example show.

Example 2. Let C be an elliptic curve with $|C(\mathbb{F}_q)| = 1 + q$ and let $D = 3P_0$. Then $\tau = 3$ and Γ is triples of collinear points on C . In this case we have

$$\begin{aligned} |X(\mathbb{F}_q)| &= |C(\mathbb{F}_q)|(|C(\mathbb{F}_q)| - 1)(|C(\mathbb{F}_q)| - 2) = q^3 - q \\ |\Gamma(\mathbb{F}_q)| &= (|C(\mathbb{F}_q)| - 9)(|C(\mathbb{F}_q)| - 1 - 4) = (q - 8)(q - 4) = q^2 - 12q + 32 \end{aligned}$$

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assuming that the 2-torsion and 3-torsion points are \mathbb{F}_q -rational. This follows from the fact that 3 points on C are collinear if and only if they have sum 0 in the group structure on the elliptic curve. Vi now have for all uneven j , that

$$|X(\mathbb{F}_{q^j})| - q |\Gamma(\mathbb{F}_{q^j})| = -12(q^j)^2 - 36q^j.$$

A result of the above type has applications in the assessment of the performance of decoding algorithms for algebraic geometry codes according to [JNH].

Central to the proof of the theorem is the following lemma, which is obtained through a geometric construction.

Lemma 3. *In the notation above*

- i) $X \setminus \Gamma$ is affine.
- ii) Γ is smooth if $\deg(D) \geq 2g + 1$

Proof. Let $(a_{i,1} : \dots : a_{i,\tau})$ be homogenous coordinates on the i 'th copy of $\mathbb{P}^{\tau-1}$ in $\mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}$ and let $V \subset \mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}$ be the closed subscheme defined by the vanishing of the determinant

$$\begin{vmatrix} a_{1,1} & \dots & a_{\tau,1} \\ a_{1,2} & \dots & a_{\tau,2} \\ \dots & \dots & \dots \\ a_{1,\tau} & \dots & a_{\tau,\tau} \end{vmatrix}$$

Consider for a moment the Segre embedding

$$\overbrace{\mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}}^{\tau\text{-fold}} \xrightarrow{\text{Segre}} \mathbb{P}^N, \quad N = \tau! - 1$$

the morphism defined by

$$(a_{1,1} : \dots : a_{1,\tau}) \times \dots \times (a_{\tau,1} : \dots : a_{\tau,\tau}) \mapsto (\dots : a_{1,i_1} \cdot a_{2,i_2} \cdot \dots \cdot a_{\tau,i_\tau} : \dots).$$

Then we see, that $V \subset \mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}$ is the inverse image of a hyperplane $H \in \mathbb{P}^N$.

By assumption $\deg(D) \geq 2g + 1$, therefore $\tau = l(D) = \deg(D) + 1 - g$ by Riemann-Roch, and the divisor D defines an embedding of the curve C as a smooth curve in $\mathbb{P}^{\tau-1}$:

$$\phi : C \rightarrow \mathbb{P}^{\tau-1}.$$

By the definition of X and Γ , we have that (P_1, \dots, P_τ) is in Γ if and only if $\phi(P_1), \dots, \phi(P_\tau)$ are linear dependent in $\mathbb{P}^{\tau-1}$, equivalently lie in a hyperplane $L \subset \mathbb{P}^{\tau-1}$, therefore we have the cartesian diagrams of intersections:

$$\begin{array}{ccccccc} X & \longrightarrow & \overbrace{C \times \dots \times C}^{\tau\text{-fold}} & \xrightarrow{\phi \times \dots \times \phi} & \overbrace{\mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}}^{\tau\text{-fold}} & \xrightarrow{\text{Segre}} & \mathbb{P}^N \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Gamma & \longrightarrow & (\phi \times \dots \times \phi)^{-1}(V) & \longrightarrow & V & \longrightarrow & H \end{array}$$

and we note the important fact that

$$X \setminus \Gamma = \overbrace{C \times \dots \times C}^{\tau\text{-fold}} \setminus (\phi \times \dots \times \phi)^{-1}(V).$$

It follows that $X \setminus \Gamma$ is isomorphic to the complement of a hyperplane section in a projective variety and therefore affine, which was the first assertion.

As for assertion on smoothness, assume to the contrary that $(P_1, \dots, P_\tau) \in \Gamma$ is a singular point on Γ , this implies that H (and thereby V) do not intersect X transversally at (P_1, \dots, P_τ) .

Let L be a hyperplane in $\mathbb{P}^{\tau-1}$ through P_1, \dots, P_τ , which exist as $(P_1, \dots, P_\tau) \in \Gamma$. All τ -tuples of points in L are linear dependent, i.e. for all j , therefore we have

$$L_j := P_1 \times \dots \times P_{j-1} \times L \times P_{j+1} \times \dots \times P_\tau \subset V \subset \mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}.$$

Consider the Cartesian diagrams of intersections in $\mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1}$:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^{\tau-1} \times \dots \times \mathbb{P}^{\tau-1} \\ \uparrow & & \uparrow \\ \Gamma & \longrightarrow & V \\ \uparrow & & \uparrow \\ P_1 \times \dots \times P_{j-1} \times L \cap C \times P_{j+1} \times \dots \times P_\tau & \longrightarrow & L_j \end{array}$$

As the intersection between X and V isn't transversal at (P_1, \dots, P_τ) , the intersection between X and $P_1 \times \dots \times P_{j-1} \times L \times P_{j+1} \times \dots \times P_\tau$ can't be either, consequently L is a tangent hyperplane to the curve C at P_j . This is true for all P_1, \dots, P_τ , i.e. , there exists a rational functions in $L(D)$ vanishing to at least second order at P_1, \dots, P_τ , therefore $l(D - (2P_1 + \dots + 2P_\tau)) > 0$, however this contradicts the assumption as

$$\begin{aligned} \deg(D - (2P_1 + \dots + 2P_\tau)) &= \deg(D) - 2l(D) \\ &= \deg(D) - 2(\deg(D) + 1 - g) \\ &= 2g - 2 - \deg(D) < 0. \end{aligned}$$

□

Assume that the prime l is different from the characteristic of the ground field. Let \mathbb{Q}_l denote the l -adic numbers. For a constructible sheaf \mathcal{F} of \mathbb{Q}_l -vector spaces $H^i(X, \mathcal{F})$ (resp. $H_c^i(X, \mathcal{F})$) denote the étale l -adic cohomology groups (resp. the étale l -adic cohomology groups with compact support), see [M].

Finally for an integer c we denote by $\mathcal{F}(c)$ the Tate twist of \mathcal{F} and

$$H^i(X, \mathbb{Q}_l(c)) = H^i(X, \mathbb{Q}_l(c)) \otimes \mathbb{Q}_l(c)$$

The second main ingredient in the proof is a Lefschetz Theorem for quasi-projective varieties. We have not been able to find a reference for it and gives a proof along the lines of [J, Corollaire 7.2], see also [G-L] for related results.

Lemma 4. A Lefschetz Theorem for quasi-projective varieties. *Let $X \subset \mathbb{P}^N$ be a quasi-projective, smooth scheme of dimension n and let $Y = X \cap H$ be a smooth hyperplane section, such that $X \setminus Y$ is affine. Then there are isomorphisms:*

$$H_c^{i-2}(Y, \mathbb{Q}_l(-1)) \rightarrow H_c^i(X, \mathbb{Q}_l)$$

for $i \geq n + 2$.

Proof. For any locally constant sheaf \mathcal{F} of $\mathbb{Z}/(l)$ -modules, the inverse image morphisms:

$$H^i(X, \mathcal{F}) \rightarrow H^i(Y, \mathcal{F}) \tag{2}$$

are isomorphisms for $i \leq n - 2$ as follows from the long exact cohomology sequence using the assumption that $X \setminus Y$ is affine. As both X and Y are assumed to be smooth, Poincaré duality applied to (2) gives the result. \square

We are ready to prove Theorem 1.

Proof. The ground field is the finite field \mathbb{F}_q and $H_c^i(X, \mathbb{Q}_l)$ is endowed with an action of the Frobenius morphism \mathbf{Frob} . The Lefschetz trace formula [M, p.292] by A. Grothendieck determines the number of \mathbb{F}_q -rational points in terms of the traces of \mathbf{Frob} on the étale cohomology spaces.

We have accordingly

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2\tau} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(X, \mathbb{Q}_l)) \quad (3)$$

$$q |\Gamma(\mathbb{F}_q)| = q \sum_{i=0}^{2\tau-2} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(\Gamma, \mathbb{Q}_l)) \quad (4)$$

As for the high dimensions, we obtain from Lemma 4 applied to X and Γ , that

$$\begin{aligned} q \sum_{i=\tau}^{2\tau-2} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(\Gamma, \mathbb{Q}_l)) &= \sum_{i=\tau}^{2\tau-2} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(\Gamma, \mathbb{Q}_l(-1))) = \\ &= \sum_{i=\tau+2}^{2\tau} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(X, \mathbb{Q}_l)) \end{aligned}$$

Combining this with (3) and (4) gives:

$$\begin{aligned} |X(\mathbb{F}_q)| - q |\Gamma(\mathbb{F}_q)| &= \\ \sum_{i=0}^{\tau+1} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(X, \mathbb{Q}_l)) &- q \sum_{i=0}^{\tau-1} (-1)^i \text{Tr}(\mathbf{Frob} | H_c^i(\Gamma, \mathbb{Q}_l)) \end{aligned}$$

Deligne's main theorem [D] gives that the eigenvalues of \mathbf{Frob} 's action on the i 'th cohomology group have absolute values $\leq q^{\frac{i}{2}}$. This immediately implies (1) of Theorem 1 as the dimensions on the cohomology groups do not depend on the power j of q and the highest power of q being $q^{\frac{\tau+1}{2}}$. \square

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