Local groupoids and etendues

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For any topological space X, there is an etale topological groupoid $\Gamma^{(X)}$ of germs of partial homeomorphisms from one point of X to another. So the space of objects is X, and an arrow from x to y is represented by a homeomorphism $f : U \to V$ where U and V are neighbourhoods of x and y, respectively:

$$\begin{array}{ccc} X & & X \\ \uparrow & & \uparrow \\ U & \xrightarrow[f]{\cong} & V \end{array}$$

The aim of this talk is to generalize this construction of $\Gamma^{(X)}$ to e.g. the localic situation, where we would like avoid using points. It turns out that large part of the notions and theory needed exist in the work of Ehresmann from the 1950's, notably in "Gattungen von lokalen Strukturen" (J.-Ber. Deutsch. Math. Verein 1957; a translation into French exists in Cahiers Vol. 3)

There are two crucial notions in loc.cit.: local class, and local groupoid.

By a *local class*, one understands a partially ordered (or preordered) set which is conditionally cocomplete (i.e., any subset with an upper bound has a least upper bound), and satisfies the distributive law

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i).$$

(where $\bigvee$ denotes least upper bound (=sup), and $\wedge$ denotes meet; note that binary meets exist in any conditionally cocomplete poset). A *local set* is a *small* local class, which is furthermore partially ordered rather than just preordered. Because of the distributive law, we see that if a local set has a maximum element 1, then it is a frame; more generally, in a local set, any principal lower set $\downarrow(a)$ is a frame.

By a *local groupoid* (Ehresmann), one understands a groupoid $\Phi_1 \Rightarrow \Phi_0$ in the category of posets, such that $d_0 : \Phi_1 \to \Phi_0$ for each $f \in \Phi_1$ induces an order isomorphism between $\downarrow (f)$ and $\downarrow d_0(f)$, and similarly for d_1; and such that Φ_1 and (hence) Φ_0 are local classes.

The partial homeomorphisms considered above form a local groupoid, the order is given by $g \leq f$ if g is a *restriction* of f. For a general local groupoid, if $U' \leq U = d_0(f)$, there is a unique $f' \leq f$ with $d_0(f') = U'$; this f' we of course may denote $f \mid U'$.

Similarly, given an object X in a topos $\underline{E}$, we get a local groupoid of partial isomorphisms of subobjects of X. We denote it $\Phi^{(X)}$. (This notion is really relative to a geometric morphism $\gamma : \underline{E} \to \underline{S}$.) Note that in these examples, the local set of objects is in fact a frame.

An alternative way to describe this local groupoid is to describe its nerve; the n'th part Φ_n is given by the local class consisting of all $n+1$-tuples of parallel monic maps with codomain X (or rather, the partially ordered set associated to this preordered class).

Finally, to any local groupoid Φ, we may associate canonically a certain quotient groupoid $S\Phi$, which is likewise a local groupoid. Namely, identify two parallel arrows $f, g : U \to V$ in Φ if they induce the same frame isomorphism $\downarrow (U) \to \downarrow (V)$ (where the frame isomorphism induced by f to $U' \leq U$ associates $d_1(f \mid U')$, and similarly for g. It is not hard to see that the quotient set carries structure of local groupoid again. The local set of objects of these two groupoids Φ and $S\Phi$ is the same.

It is possible to associate directly to a local groupoid Φ a topos $B\Phi$ of sheaves on which it acts; in fact, I believe that, as a category, $B\Phi$ may be described as the category of *complete species of local structures* over Φ in the sense of Ehresmann (loc.cit.) (species of local structures = Gattung von lokalen Strukturen). However, I am not sure, and anyway, it is better to relate the concepts directly to those which today are so well elaborated, namely locales, and (etale) localic groupoids.

First, the relationship between frames and local sets.

1) A local set with a maximal element is the same thing as a frame.

2) Any lower set in a local set is a local set. In particular, any lower set in a frame is a local set.

3) Any local set appears as a lower set in a frame; in fact, one may prove

Proposition 1 *To any local set A, there is a unique (up to isomorphism)* *frame $\overline{A}$ which contains A as a lower set and such that the elemnts of A sup-generate $\overline{A}$. The inclusion $A \subseteq \overline{A}$ preserves sup.*

(We call $\overline{A}$ the *frame completion* of A. It may be constructed as $C - Idl(A)$, where C is the canonical site structure (coverage) on A.) The locale associated to a frame O, we denote $\overline{O}$. So the locale associated to the frame completion of a local set Φ, we are forced to denote $\overline{\overline{\Phi}}$.

Given an open map between locales $p : E \to B$, recall that an element $a \in O(E)$ is called p-small if p^+ maps $\downarrow (a)$ bijectively to $\downarrow (p^+(a))$, where p^+ is the left adjoint of $p^{-1} : O(B) \to O(E)$. The p-small elements form a lower set in $O(E)$, hence a local set. To say that p is etale is equivalent to saying that $O(E)$ is the frame completion of the lower set of p-small elements, or, equivalently, the frame completion of *some* lower set of p-small elements.

Conversely, if $q : \Phi \to \Psi$ is an order preserving map between local sets, such that for any $a \in \Phi$, q maps $\downarrow (a)$ order isomorphic to $\downarrow (q(a))$, (i.e., q is a discrete fibration of posets), then there is a unique etale map $p : \overline{\overline{\Phi}} \to \overline{\overline{\Psi}}$ with p^+ agreeing with q on the elements of $\Phi \subseteq \overline{\Phi} = O(\overline{\overline{\Phi}})$.

To be able to describe the relationship between etale localic groupoids and local groupoids, we need to understand those pull-backs of locales which necessarily come up. Pull backs of locales generally are nasty, but certain ones are very easy. I would imagine that the following "pull back lemma" is well known: namely that certain pull-backs may be calculated as pull-backs of posets (in fact pull-backs of local sets), rather than as pushout of frames. More precisely:

Proposition 2 *Let there be given a pull-back diagram of locales,*

$$\begin{array}{ccc} P & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

with g (and hence h) a surjective etale map, and with f (and hence k) an open map. Let $s(Y)$ be a sup generating lower set of g-small elements in

$O(Y)$, and let $s(X)$ be a sup-generating lower set in $O(X)$. Then the frame $O(P)$ is the frame completion of the local set obtained as the pull-back (in the category of posets) of $f^+ : s(X) \to O(Z)$ and $g^+ : s(Y) \to O(Z)$. (To (a,b) with $f^+(a) = g^+(b)$ corresponds the element $h^{-1}(a) \wedge k^{-1}(b) \in O(P)$.)

For the application to the construction of an etale localic groupoid out of a local groupoid, both f and g are etale, namely the etale locale maps associated (by Prop) to the discrete fibrations of local sets d_0 and $d_1 : \Phi_1 \to \underline{\Phi_0}$. The localic groupoid we construct has its G_1 and G_0 equal to $\overline{\overline{\Phi_1}}$ and $\overline{\overline{\Phi_0}}$, respectively. By the pull-back lemma, the pull-back

$$G_1 \times_{G_0} G_1$$

is the locale associated to the frame completion of the pull-back of posets (local sets)

$$\Phi_1 \times_{O(\overline{\overline{\Phi_0}})} \Phi_1.$$

And to get the desired (etale) 'composition' map

$$G_1 \times_{G_0} G_1 \to G_1,$$

it suffices by Proposition 1 to give a suitable map between the generating local sets, and this map is taken to be the composition that makes up the groupoid structure of Φ,

$$\Phi_1 \times_{\Phi_0} \Phi_1 \to \Phi_1.$$

For completeness, let us describe how one, conversely, may directly construct a local groupoid Φ out of an etale localic groupoid $G_1 \Rightarrow G_0$. Namely, take $\Phi_0 = O(G_0)$, and take Φ_1 to be the lower set in $O(G_1)$ consisting of those s which are d_0-small as well as d_1-small. Composition of such 'doubly-small' elements is achieved by identifying s by a section of d_0 over an open sublocale of G_0.

Let me finish by indicating how the explicit description of the local groupoid $\Phi^{(X)}$ of partial isomorphisms of an object X in a topos $\underline{E}$, (with associated etale localic groupoid $\Gamma^{(X)}$), gives rise to a comparison functor

$$\underline{E} \to B\Gamma^{(X)} \simeq B\Phi^{(X)}.$$

For the case where X is fully supported and $\underline{E}/X$ is localic (thus $\underline{E}$ an etendue, with X as witnessing object, we shall prove that this comparison is an equivalence; its role (universal property, dependence of the choice of X) without the localicness assumption, has not yet been investigated, so far I know.

We already mentioned how the object$X$ gives rise to a simplicial object in the category of local classes (and discrete fibrations between them): the n'th part $\Phi_n^{(X)}$ is given by the local class consisting of all $n+1$-tuples of parallel monic maps with codomain X. It is a local class, - preordered, but if we take the associated partial order, we get a local set. Since the face maps are discrete fibrations of posets, we get by Proposition 1 that the induced maps between the associated locales are in fact etale, so we have an etale simplicial locale $G_\bullet$ with $O(G_0) =$ subobject frame of $X = \Phi_0^{(X)}$ and $O(G_1) = \overline{\Phi}_1^{(X)}$. In fact, it is clear that $\Phi_n^{(X)}$ is the n-fold pull-back of the two structural maps $\Phi_1^{(X)} \to \Phi_0^{(X)}$ (in the category of posets), and from the pull-back lemma, it therefore follows that the associated simplicial locale $G_\bullet$ is in fact the nerve of the etale localic groupoid Γ^X.

To describe the canonical 'comparison' geometric morphism $\underline{E} \to B\Phi^{(X)}$, we utilize that both toposes here appear as descent toposes for simplicial toposes, namely, those which in degree n have, respectively, $\underline{E}/X^{n+1}$ and $sh(M_{n+1})$ (M_{n+1} denoting the locale associated to the frame completion of the local set $\Phi_n^{(X)}$ which in turn is the poset associated to the local class M^*_{n+1} of $n+1$-tuples of parallel monic maps with codomain X.). We have an easy comparison geometric morphism for each n

$$\underline{E}/X^{n+1} \to sh(M_{n+1});$$

in fact, we have an evident functor

$$M^*_{n+1} \to \underline{E}/X^{n+1} \tag{1}$$

taking an $n+1$-tuple of monics $f_i : U \to X$ to the object $< f_1, \ldots, f_{n+1} >: U \to X^{n+1}$, and this functor preserves coverings.

In case $\underline{E}/X$ is localic, then so is $\underline{E}/X^{n+1}$, and therefore

$$\underline{E}/X^{n+1} \simeq sh(P(X^{n+1}))$$

where $P(Y)$ denotes the subobject lattice (frame) of Y. So in the case where $\underline{E}/X$ is localic, the above comparisons are

$$sh(P(X^{n+1})) \to sh(M_{n+1}).$$

They are in fact induced by an evident inclusion (described by the same formula as (**??**) above)

$$M_{n+1} \to P(X^{n+1})$$

which identifies the left hand side with a lower set on the right. As such, it is a sup-generating subset: for, $\underline{E}/X$ being localic is equivalent to saying that all maps into X are *locally* monic in the sense of [Kock-Moerdijk:Presentations of Etendues, Bangor Volume (Cahiers)]. But this implies that the domain of any monic $g : U \to X^{n+1}$ may be covered by subobjects V's such that the restriction of g to each V has the property that each composite $V \to U \to X^{n+1} \to X$ is monic (where the last map is projection to a factor). This clearly implies the sup-genaration assertion. Thus the induced geometric norphism is an equivalence. It follows that the descended comparison $\underline{E} \to B\Phi^{(X)}$ is an equivalence.