Infinitesimal Deformations of Complete Vector Fields are Complete*

Anders Kock

Recall that a vector field X on a smooth manifold M is *complete* if it generates a global flow $\xi : M \times \mathbb{R} \to M$. If X_ϵ is a smoothly parametrized family of vector fields ($\epsilon \in \mathbb{R}$), it may happen that X_0 is complete but all X_ϵ with $\epsilon \neq 0$ are incomplete. This is the case for instance with the family of vector fields

$$\epsilon \cdot x^2 \cdot \frac{\partial}{\partial x}$$

on $\mathbb{R}$.

We shall prove, however, that in the context of synthetic differential geometry, if X_0 is complete, then so is X_ϵ for any ϵ with $\epsilon^2 = 0$.

We presuppose some general notions from synthetic differential geometry (SDG), cf. [3], but also some more particular results [5]. The notion 'connection in a groupoid', 'deplacement' and 'deplacement field' that we use, are essentially due to Ehresmann, cf. [1].

The results in the present note were presented at the PSSL meeting in Amsterdam, June 1984.

1 Generalities concerning vector fields and flows

Let D be an object equipped with a $0 \in D$ and an inversion 'minus' $D \to D$. An *action* of D on an object M is a map

$$X : M \times D \to M$$

*This is a retyping of Aarhus Math. Preprint 1985/86 No. 23 (February 1986).

with

$$(1.1) \qquad X(m,0) = m = X(X(m,d), -d)$$

$\forall m \in M$, $\forall d \in D$. We shall work in the context of SDG, with $D = \{d \in D \mid d^2 = 0\}$; then X is a vector field on M, (this conception is due to Lawvere). If M is infinitesimally linear, the second equation in (1.1) follows from the first.

If (E,Y) and (M,X) are objects equipped with vector fields, we say that a map $\pi : E \to M$ is a *morphism of vector fields* if $\pi(Y(c,d)) = X(\pi(c),d)$ $\forall c \in E$ $\forall d \in D$; one also says that the 'vector field Y is *projectable* to the vector field X *via* π '.

For such π, we get for each $m \in M$ and $d \in D$ an invertible mapping (a 'fibre infinitesimal transformation') from E_m, the fibre over m, to $E_{X(m,d)}$, the fibre over $X(m,d)$, namely the restriction of $Y(-,d) : E \to E$ to E_m. This map $E_m \to E_{X(m,d)}$ may be viewed as an arrow $m \to X(m,d)$ in the groupoid $\mathrm{Full}(\pi)$ consisting of invertible maps between the fibres of π. If $\Psi \subset \mathrm{Full}(\pi)$ is a subgroupoid, it makes sense to ask that all the maps

$$Y(-,d) : E_m \to E_{X(m,d)}$$

belong to the subgroupid Ψ, in which case we shall say that the morphism π of vector fields is *subordinate* to Ψ. (In the terminology of §4 below, the fibre infinitesimal transformations form a "deplacement field" in the groupoid Ψ.)

The line R in SDG carries a canonical vector field $\frac{\partial}{\partial x}$, namely $(x,d) \mapsto x + d$ for $x \in R$, $d \in D$. We let $I \subset R$ be any subset containing 0 and stable under the action of $\frac{\partial}{\partial x}$, e.g. $I = R$, $I = [0,1]$ or $I = D_\infty$. We shall assume that the vector fields (M,X) we consider have the property that for any $m \in M$, there is at most one morphism of vector fields

$$(I, \frac{\partial}{\partial x}) \to (M,X)$$

with $0 \mapsto m$. Such a morphism is adequately called a *streamline* for X through m.

For $s,t \in I$, we write $s \sim t$ if $s - t \in D$. This is a reflexive symmetric relation.

Let $I_{(1)} \subset I \times I$ be the set of pairs (s,t) with $s \sim t$. Let G be a group. A G-valued 1-form on I is a map

$$\omega : I_{(1)} \to G$$

with $\omega(t,t) = e$ and $\omega(s,t) = \omega(t,s)^{-1}$ $\forall s \sim t$. We say that the group G *admits integration* if for any such 1-form ω, there exists a unique $f : I \to G$ with $f(0) = e$ and

$$\omega(s,t) = f(t) \cdot f(s)^{-1} \quad \forall s \sim t.$$

This holds in the models of SDG if G is a Lie group and I an interval on R, cf. [6] and [4], but fails for large groups like $\mathrm{Diff}(R)$ (= the object of bijective maps $R \to R$, cf. [4]), a fact closely related to the existence of incomplete vector fields.

Recall (from [5], say) that a groupoid Ψ over M is called *transitive* if for any $m, n \in M$, there is at least one arrow $m \to n$ in Ψ. In this case, all the isotropy groups $\Psi(m,m)$ are isomorphic.

Theorem 1.1 *Let $\pi : (E,Y) \to (M,X)$ be a morphism of vector fields, subordinate to a groupoid $\Psi \subset \mathrm{Full}(\pi)$, and assume Ψ is transitive and that its isotropy groups admit integration. Let $c \in E$. Then any streamline $\xi : I \to M$ for X through $\pi(c)$ lifts to a streamline η for Y through c.*

Proof. Let $\xi^*(\Psi)$ be the full image of the groupoid Ψ along ξ (i.e. the hom set $\xi^*(\Psi)(s,t)$ can be identified with the hom set $\Psi(\xi(s), \xi(t))$ of Ψ). We get a connection ∇ ([5]) on the groupoid $\xi^*(\Psi)$, namely

$$\nabla(t, t+d) := Y(-, d) \mid_{E_{\xi(t)}} : E_{\xi(t)} \to E_{\xi(t+d)}$$

which by assumption is an arrow $\xi(t) \to \xi(t+d)$ in Ψ, hence an arrow in $\xi^*(\Psi)$ from t to $t+d$.

We now apply Theorem 4.1 of [5], with the identity map $I \to I$ as f, to get a functor

$$\overline{\nabla} : I \times I \to \xi^*(\Psi)$$

over I extending ∇ (where $I \times I = \pi_0 I$ is the codiscrete groupoid over I); $\overline{\nabla}$ satisfies

$$\nabla(t, t+d) \circ \overline{\nabla}(s,t) = \overline{\nabla}(s, t+d)$$

$\forall s, t \in I$ $\forall d \in D$.

Now define $\eta : I \to E$ by

$$\eta(s) := \overline{\overline{\nabla}}(0,s)(c),$$

where $\overline{\overline{\nabla}} : I \times I \to \Psi$ is the funcor obtained from $\overline{\nabla}$ by composing with the canonical functor $\xi^*\Psi \to \Psi$. Then $\pi \circ \eta = \xi$, for

$$\pi(\eta(s)) = \partial_1(\overline{\overline{\nabla}}(0, s)) = \xi(s)$$

since the codomain of $\overline{\nabla}(0, s)$ is s. So η lifts ξ. Also clearly $\eta(0) = c$. To see that η is a streamline for Y, note that we have

$$\begin{aligned} \eta(s + d) &= \overline{\overline{\nabla}}(0, s + d)(c) \\ &= \overline{\overline{\nabla}}(s, s + d) \circ \overline{\overline{\nabla}}(0, s)(c) \quad (\overline{\overline{\nabla}} \text{ being a functor}) \\ &= \overline{\overline{\nabla}}(s, s + d)\eta(c) \\ &= Y(-, d)(\eta(c)), \end{aligned}$$

since

$$\overline{\overline{\nabla}}(s, s + d) = \overline{\nabla}(s, s + d) = \nabla(s, s + d),$$

which is the fibre infinitesimal transformation $Y(-, d) : E_{\xi(s)} \to E_{\xi(s+d)}$.

2 Reduction to the fibrewise affine groupoid

We consider as in §1 a vector field $X : M \times D \to M$ on an object M, and a deformation H of it, with parameter space for the deformation a pointed object $(D', 0)$ (which ultimately will be taken to be D). Thus, the data we consider is a map

$$M \times D \times D' \xrightarrow{H} M$$

with

$$H(m, d, 0) = X(m, d) \quad \forall m \in M,\ d \in D \tag{2.1}$$

and

$$H(m, 0, \epsilon) = m \quad \forall m \in M,\ \epsilon \in D'. \tag{2.2}$$

We write X_ϵ for $H(-, -, \epsilon)$. Thus $X_0 = X$, by (2.1), and X_ϵ is a vector field by (2.2), provided M is infinitesimally linear, which we henceforth assume (in fact in the 'strong' sense, cf. [7]).

Let $M^{D'}$ denote the set of maps $D' \to M$. We equip this set with a vector field

$$\widetilde{H} : M^{D'} \times D \to M^{D'}$$

given by

$$\widetilde{H}(\tau, d)(\epsilon) := X_\epsilon(\tau(\epsilon), d) = H(\tau(\epsilon), d, \epsilon)$$

for $\tau \in M^{D'}$, $d \in D$, $\epsilon \in D'$. (Note that by (2.2)

$$\widetilde{H}(\tau, 0)(\epsilon) = H(\tau(\epsilon), 0, \epsilon) = \tau(\epsilon),$$

so $\widetilde{H}(\tau, 0) = \tau$, so $\widetilde{H}$ is indeed a vector field, $M^{D'}$ being infinitesimally linear since M is).

Consider, for fixed $\epsilon \in D'$, the map 'evaluation at ϵ'

$$M^{D'} \xrightarrow{\mathrm{ev}_\epsilon} M. \tag{2.3}$$

We note that this map is a morphism of vector fields

$$(M^{D'}, \widetilde{H}) \to (M, X_\epsilon);$$

for

$$\begin{aligned} \mathrm{ev}_\epsilon(\widetilde{H}(\tau, d)) &= \widetilde{H}(\tau, d)(\epsilon) \\ &= X_\epsilon(\tau(\epsilon), d) \\ &= X_\epsilon(\mathrm{ev}_\epsilon(\tau), d). \end{aligned}$$

Let us denote the map $\mathrm{ev}_0 : M^{D'} \to M$ by π; so in particular

$$(M^{D'}, \widetilde{H}) \xrightarrow{\pi} (M, X) \tag{2.4}$$

is a morphism of vector fields.

We now take $D' = D$, so $M^{D'} = M^D \xrightarrow{\pi} M$ is the tangent bundle of M. Since M is infinitesimally linear, the fibres of this map are R-modules. So it makes sense to talk about R-linear and R-affine maps from one fibre to another (affine = linear + constant).

Lemma 2.1 *The fibre infinitesimal transformations of the vector field morphism (2.4) (for $D = D'$) are affine maps.*

Proof. Let $m \in M$, $d \in D$, and consider the corresponding fibre infinitesimal transformation

$$(M^{D'})_m \xrightarrow{\phi} (M^{D'})_{X(m,d)}. \tag{2.5}$$

Let $\gamma \in (M^{D'})_{X(m,d)}$ be given by

$$\gamma(\epsilon) = H(m, d, \epsilon).$$

We prove that ϕ minus the constant γ is a linear map, i.e. that the map ϕ_0 defined by

$$\phi_0(\tau) := \phi(\tau) - \gamma$$

depends in a linear way on τ, so that ϕ itself is affine. Now since M is infinitesimally linear, the R-modules $(M^D)_n$ satisfy the 'vector form of the Kock-Lawvere axiom', as observed by Lavendhomme [9], so that linearity of ϕ_0 is equivalent, by [3] Prop. I.10.2, to homogeneity,

$$\phi_0(\lambda \cdot \tau) = \lambda \cdot \phi_0(\tau) \quad \forall \lambda \in R.$$

So consider $\phi(\tau) - \gamma \in (M^{D'})_{X(m,d)}$. To form this difference, we use the standard recipe for adding tangent vectors on infinitesimally linear objects (cf. [3]), namely 'construct a map $D(2) \to M$ and restrict it along the diagonal'. In the present case, we construct a map $l : D(2) \to M$ given by

$$l(\epsilon_1, \epsilon_2) := H(\tau(\epsilon_1), d, \epsilon_1 - \epsilon_2)$$

($d \in D$ being fixed, and $\epsilon_1 - \epsilon_2 \in D$ since $(\epsilon_1, \epsilon_2) \in D(2)$.) We clearly have

$$l(\epsilon, 0) = H(\tau(\epsilon), d, \epsilon) = \phi(\tau)(\epsilon)$$

and

$$l(0, \epsilon) = H(\tau(0), d, -\epsilon) = (-\gamma)(\epsilon)$$

so that

$$l(\epsilon, \epsilon) = (\phi(\tau) - \gamma)(\epsilon)$$

or

$$\begin{aligned} \phi_0(\tau)(\epsilon) &= (\phi(\tau) - \gamma)(\epsilon) \\ &= H(\tau(\epsilon), d, \epsilon - \epsilon) = H(\tau(\epsilon), d, 0). \end{aligned}$$

We thus have, for $\lambda \in R$

$$\begin{aligned}\phi_0(\lambda\tau)(\epsilon) &= H((\lambda \cdot \tau)(\epsilon), d, 0)\\ &= H(\tau(\lambda \cdot \epsilon), d, 0)\\ &= \phi_0(\tau)(\lambda \cdot \epsilon)\\ &= (\lambda \cdot \phi_0(\tau))(\epsilon),\end{aligned}$$

proving homogeneity of ϕ_0, thus proving ϕ affine.

3 The theorem

We now assume that M is a manifold, of dimension n, say. This implies that the tangent bundle is a vector bundle, so that the groupoid of fibrewise linear maps is transitive. Then the groupoid Ψ of fibrewise affine maps is transitive as well. The isotropy-groups of Ψ are Lie groups, namely the n^2+n dimensional group of affine maps $R^n \to R^n$. So they admit integration. If $H : M \times D \times D' \to M$ is a deformation of a vector field, as in §1, we therefore get, by Lemma 2.1 and Theorem 1.1 that the morphism (2.4) of vector fields has the property that streamlines for X lift to streamlies for $\widetilde{H}$; we can then prove:

Theorem 3.3 *Let M be a manifold, and let $\{X_\epsilon \mid \epsilon \in D\}$ a deformation of a complete vector field $X = X_0$. Then each X_ϵ is complete.*

Proof. Let $m \in M$, and let $\epsilon \in D$ be fixed. By assumption, there is a streamline $\xi : I \to M$ for X through m. By the above remarks, it lifts to a streamline η for $\widetilde{H}$ through the element $\tau : D \to M$ with constant value m. Compose η with $\mathrm{ev}_\epsilon : M^D \to M$ which is a morphism of vector fields $(M^D, \widetilde{H}) \to (M, X_\epsilon)$; so we get a morphism of vector fields

$$(I, \frac{\partial}{\partial x}) \xrightarrow{\eta} (M^D, \widetilde{H}) \xrightarrow{\mathrm{ev}_\epsilon} (M, X_\epsilon)$$

through $\mathrm{ev}_\epsilon(\eta(0)) = \mathrm{ev}_\epsilon(\tau) = m$. Since thus through each m, there is a streamline for X_ϵ, X_ϵ is a complete vector field.

4 Complete deplacement fields

The argument of §1 could have been presented in greater generality by the introduction of the notion of 'deplacement' and 'deplacement field'. We sketch how this would go.

Let Ψ be a groupoid over M. A *deplacement at* $m \in M$ is a map $g : D \to \Psi$ satisfying

$$g(0) = \mathrm{id}_m$$

$$\partial_0(g(d)) = m \quad \forall d \in D.$$

A *deplacement field* G on Ψ is a law which to each $m \in M$ associates a deplacement at m, so $G : M \times D \to \Psi$ satisfies

$$G(m, 0) = \mathrm{id}_m \quad \forall m \in M$$

$$\partial_0(G(m, d)) = m \quad \forall m \in M \; \forall d \in D.$$

If G is a deplacement field on Ψ, we get a vector field $\partial_1 G$ on M by

$$(\partial_1 G)(m, d) = \partial_1(G(m, d)).$$

Inspecting the proof of Theorem 1.1, we see that we essentially might as well have proved

Theorem 4.1 *Assume that* Ψ *is transitive and that its isotropy groupoids admit integration. Then if* $\partial_1 G$ *is a complete vector field on* M*,* G *itself is a complete displacement field on* Ψ *in the sense that there exists a map*

$$\overline{G} : M \times I \to \Psi$$

satisfying

$$\overline{G}(m, 0) = \mathrm{id}_m \quad \forall m \in M$$

$$\partial_0(\overline{G}(m, d)) = m \quad \forall m \in M \; \forall d \in D$$

and

$$\overline{G}(m, s + d) = G(\partial_1(\overline{G}(m, s), d)) \circ \overline{G}(m, s)$$

$\forall m \in M \; \forall s \in I \; \forall d \in D.$

Proof (sketch). Let $\xi : I \to M$ be a streamline for $\partial_1 G$ through m. As in the proof of Theorem 1.1, obtain a functor $\overline{\overline{\overline{\nabla}}} : I \times I \to \Psi$ over ξ; then

$$\overline{G}(m, s) := \overline{\overline{\overline{\nabla}}}(0, s)$$

will satisfy the conditions.

This result may be seen as a paraphrasing of the Corollary 1 of §33 in [8],

For the viewpoint of a deplacement field as an action on Ψ by a small pointed groupoid $\mathbb{D}$, see [2].

References

[1] C. Ehresmann, Oeuvres Vol I, Amiens 1985.

[2] A. Kock, Formal manifolds and synthetic theory of jet bundles, Cahiers de Top. et Géom. Diff. 21 (1980), 227-246.

[3] A. Kock, Synthetic Differential Geometry, Cambridge Univ. Press 1981.

[4] A. Kock, Differential forms with values in groups, Bull. Austral. Math. Soc. 25 (1982), 357-386.

[5] A. Kock, Combinatorial notions relating to principal fibre bundles, J. Pure Appl. Alg. 39 (1986), 141-151.

[6] A. Kock, Lie group valued integration in well.adapted toposes, Aarhus Preprint Series 1985/86 No. 3. [Bull. Austral. Math. Soc. 34 (1986), 395-410.]

[7] A. Kock and R. Lavendhomme, Strong infinitesimal linearity, with applications to strong difference and affine connections, Cahiers de Topologie et Géom. Diff. Catégoriques 25 (1984), 311-324.

[8] A. Kumpera and D. Spencer, Lie Equations, I, Annals of Math. Studies Vol. 73, Princeton Univ. Press 1972.

[9] R. Lavendhomme, Note sur l'algèbre de Lie d'un groupe de Lie en géométrie différentielle synthétique, Rapport no. 111, Louvain-la-Neuve 1981.