The maximal atlas of a foliation

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We shall describe such maximal atlas and provide it with an algebraic structure that brings along the holonomy groupoid almost for free. This will, at the same time, correct an attempt I made in [5] to "algebraicize" some considerations of Haefliger. - The content is not very original; one may say that it only synthesises insights that are more or less explicit in the work of Haefliger, Pradines, and Moerdijk.

1 Maximal atlas, isonomy, and holonomy

Let A be a manifold with a foliation $\mathcal{F}$ of codimension q. The maximal atlas X we are talking about has for its points the germs of $\mathcal{F}$-distinguished submersions $A \to \mathbf{R}^q$. The $\mathcal{F}$-distinguished submersions are, as usual, those submersions $f : U \to \mathbf{R}^q$ (with $U \subseteq A$) which locally have $\mathcal{F}$ as their kernel pair; or, equivalently, those smooth maps $U \to \mathbf{R}^q$ which for each $a \in U$, $df_a : T_a \to \mathbf{R}^q$ has $T_a(\mathcal{F}) \subseteq T_a A$ as its kernel.

When we henceforth say "distinguished germ", we mean a germ of an $\mathcal{F}$-distinguished submersion. The crucial algebraic property which the collection of distinguished germs have, is that if f and g are distinguished germs at a ($a \in A$), then there is a unique germ λ of a diffeomorphism of $\mathbf{R}^q$ from $f(a)$ to $g(a)$ making the following triangle commute in the category of germs of smooth maps:

$$\begin{array}{ccc} & a & \\ {}^{f_a}\swarrow & & \searrow{}^{g_a} \\ f(a) & \xrightarrow[\lambda]{\circ} & g(a) \end{array} \qquad (1)$$

(This follows from the fact that, locally around a, f and g are surjections with the *same* kernel pair, namely $\mathcal{F}$.)

Conversely, post-composing a distinguished germ f_a with a germ λ of a diffeormorphism of $\mathbf{R}^q$ clearly gives again a distinguished germ. Now the germs λ of diffeomorphisms of $\mathbf{R}^q$ form a differentiable groupoid Γ^q with $\mathbf{R}^q$ for its space of objects, and the existence and uniqueness of the $\lambda \in \Gamma$ in the above triangle means that the set X of distinguished germs is a *principal bundle* over this groupoid. This kind of structure in foliation theory was first considered by Haefliger [2]. (The classifying geometric morphism of the foliation, $c_{\mathcal{F}} : A \to \mathbf{B}(\Gamma^q)$, considered by Kock and Moerdijk (cf. [11], Theorem 5.8.1, or [6], Theorem 1) correspond to this "principal bundle", by the correspondence of Bunge [1], and Moerdijk.)

We are really just elaborating on the algebraic aspects of this structure, and in particular, utilizing the Ehresmann construction of *conjugate* action of a principal action. Also, we shall utilize the factorization (well known in topos theory, cf [3], or the Appendix) of certain smooth maps into a connected part followed by an etale part; this "connectedness" factorization is what was missing in [5] .

To bring out the algebra more clearly, we reformulate the notion of principal bundle over a groupoid, by the notion of *pregroupoid* (in the sense of [5] , not in the sense of Janelidze); we shall recapitulate this notion below. In the pregroupoid set-up, the principal bundle X (= the pregroupoid) comes *before* the groupoid that acts principally on it, and the action and the conjugate action are *derived* from X, by identical procedures. The fact that X comes before the groupoids is convenient in the analysis we shall perform, since the passage to holonomy groupoid is going to involve consideration of the *same* X (essentially), but *different* (pairs of) groupoids.

The notion of pregroupoid makes sense in any category with finite limits, for simplicity we describe it in the category of sets: a pregroupoid consists of sets X, A, B with maps $\alpha : X \to A$ and $\beta : X \to B$ together with a partially defined ternary operation $zx^{-1}y$, with $zx^{-1}y$ defined whenever $\alpha(x) = \alpha(z)$ and $\beta(x) = \beta(y)$; and then $\alpha(zx^{-1}y) = \alpha(y)$ and $\beta(zx^{-1}y) = \beta(z)$. There are equations that are assumed to hold, namely all those that hold in all groups, if $zx^{-1}y$ is interpreted as the notation suggests. The "logo" for keeping track of the book-keeping is

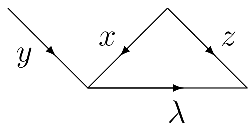
$$\begin{array}{ccc} z & \text{———} & zx^{-1}y \\ \| & & \| \\ x & \text{———} & y \end{array}$$

where single lines mean: "same β value", double lines mean "same α value".

For the maximal atlas X considered above, B is $\mathbf{R}^q$, A is (the set of points of) the manifold A, and if $x \in X$ is a distinguished germ at $a \in A$, $\alpha(x) = a$ and $\beta(x) = x(a)$. If $\alpha(x) = \alpha(z)$, there is a unique germ λ as in the figure (1) above, and then since $\beta(x) = \beta(y) = d_0(\lambda)$, $\lambda \circ y$ makes sense and is a distinguished germ from $\alpha(y)$ to $\beta(z)$; we put

$$zx^{-1}y := \lambda \circ y;$$

in picture



It is tempting, and in some sense correct, to denote the unique germ λ in this figure by $z \circ x^{-1}$, so that the defining equation for the ternary operation defining the pregroupoid structure on X reads

$$zx^{-1}y = z \circ x^{-1} \circ y.$$

This leads to another prime example of a pregroupoid: consider a groupoid Φ and two subsets A and B of the set of objects of Φ. Let X be the set of those arrows in Φ with domain in A and codomain in B, and let α and β be domain- and codomain formation, respectively. Let $zx^{-1}y := z \circ x^{-1} \circ y$. It is possible to prove that any pregroupoid arises this way. This implies that one may calculate with distinguished germs as if they were arrows in a groupoid, i.e invertible ! We shall not do this here, however.

Because of this prime example of pregroupoids, we may as well call the maps $X \to A$ and $X \to B$ the "domain" and the "codomain" map, respectively.

The name pregroupoid was chosen because out of any pregroupoid X on A, B (in the category of sets) one may construct two "edge" groupoids, $X^{-1}X \rightrightarrows A$ and $XX^{-1} \rightrightarrows B$ that act principally on X from the right and from the left, respectively, and with the two actions commuting. This was described in [5] , and is really the Ehresmann theory of conjugate actions. Likewise in [5] it was pointed out that similar constructions are possible in an arbitrary category with finite limits, provided α and β are effective descent maps. The description in the set case is quite simple: an arrow in $X^{-1}X$ from $a_1 \in A$ to $a_2 \in A$ is represented by a pair of elements $x, y \in X$ with $\beta(x) = \beta(y)$ and $\alpha(x) = a_1, \alpha(y) = a_2$. The element thus represented is denoted $x^{-1}y$, and $x^{-1}y = z^{-1}u$ iff $u = zx^{-1}y$. One may thus think of the arrows of $X^{-1}X$ as (left) fractions of elements of X. The right action on X by $X^{-1}X$ is given by $z \cdot x^{-1}y = zx^{-1}y$. Similarly, the arrows of XX^{-1} are "fractions" zx^{-1}, with $\alpha(x) = \alpha(z)$, and the left action of XX^{-1} on X is given by $zx^{-1} \cdot y = zx^{-1}y$.

Returning to the maximal atlas X for the foliation $\mathcal{F}$ on A, we described it as a pregroupoid over A and B ($B = \mathbf{R}^q$). This structure is not just a set theoretic one, but is actually smooth. First, it is clear that X, as a set of germs of functions on A, may naturally be topologized, making the map $\alpha : X \to A$ etale, and hence also providing X with the structure of smooth manifold (since A carries such structure); it is non-Hausdorff, though. With this manifold structure, the map $\beta : X \to B = \mathbf{R}^q$ becomes a submersion. It suffices to see this for a small open set of X, given as the set of germs of a distinguished submersion $f : U \to B$. But on such a small open set, β is really the same as f, hence a submersion. Also clearly the algebraic structure $zx^{-1}y$ is smooth, and since surjective submersions (and in particular etale surjections) are effective descent maps, the general construction of the groupoids $XX^{-1} \rightrightarrows A$ and $XX^{-1} \rightrightarrows B$ here gives rise to smooth groupoids; the latter is actually (isomorphic to) Γ^q itself.

The groupoid $X^{-1}X \rightrightarrows A$ has its domain- and codomain maps surjective submersions (because locally they are like $\beta : X \to B$), and it is therefore a well behaved one, in so far as categorical constructions go. I believe Pradines once mentioned the name *isonomy groupoid* of $\mathcal{F}$ for it. It is (via X) "Morita equivalent" to the groupoid $XX^{-1} = \Gamma^q$, meaning that these groupoids have equivalent toposes of sheaves-with-left-action (classifying topos) $\mathbf{B}(X^{-1}X) \simeq \mathbf{B}(XX^{-1})$ (this follows from general theory of pregroupoids, cf. [5] , and Section 2 below.). In particular, the topos $\mathbf{B}(X^{-1}X)$ does not encode the transverse structure of "A modulo $\mathcal{F}$" any more than $\mathbf{B}(\Gamma^q)$ does.

The following may be taken as a definition of the holonomy groupoid of $\mathcal{F}$, and it has the virtue of being natural (no choices of transversals or charts), and staying completely within the category of manifolds. We have noted that the map $\beta : X \to B = \mathbf{R}^q$ is a surjective submersion, hence a locally connected submersion. It may therefore be factored, in the category of manifolds,

$$X \xrightarrow{\gamma} C \to B$$

with the first map γ being connected and locally connected (a submersion with connected fibres, in fact), and the second being etale. The fibres of the map $\gamma : X \to C$ are the connected components of the fibres of the total map $\beta : X \to B$ (see the Appendix, where we summarize this factorization theory).

Proposition 1 *The pregroupoid structure on $A \leftarrow X \to B$ restricts to a pregroupoid structure on $A \leftarrow X \to C$.*

Proof. The condition for $zx^{-1}y$ to be defined is now the condition

$$\gamma(x) = \gamma(y) \text{ and } \alpha(x) = \alpha(z) \tag{2}$$

which is stronger than the previous

$$\beta(x) = \beta(y) \text{ and } \alpha(x) = \alpha(z),$$

so the problem is now just whether for the output $zx^{-1}y$, the stronger condition $\gamma(zx^{-1}y) = \gamma(z)$ holds as well, assuming (2). Now $\gamma(x) = \gamma(y)$ means that x and y are in the same component of their (common) β-fiber, and hence so are $zx^{-1} \cdot x = z$ and $zx^{-1} \cdot y = zx^{-1}y$, since left multiplication by $zx^{-1} \in \Gamma^q$ is a homeomorphism from one β fibre to another one. The verification of the equations for pregroupoid structure is trivial, since the new ternary operation $zx^{-1}y$ on X is a restriction of the old one.

We are now forced to make some decorations on the symbols XX^{-1}, $X^{-1}X$, since they refer to different "book-keeping" data: $A \leftarrow X \to B$ and $A \leftarrow X \to C$, respectively; we shall write B or C as a lower index to indicate which book-keeping is referred to. So we now have four groupoids

$$XX_B^{-1} \rightrightarrows B = \Gamma^q \rightrightarrows \mathbf{R}^q$$

$$X_B^{-1}X \rightrightarrows A \text{ (the isonomy groupoid)}$$

$$XX_C^{-1} \rightrightarrows C$$

$$X_C^{-1}X \rightrightarrows A,$$

and this latter we want to call the *holonomy* of the foliation $\mathcal{F}$ on the manifold A. (The third groupoid here, $XX_C^{-1} \rightrightarrows C$, is candidate for the title *etale* holonomy groupoid of $\mathcal{F}$, and probably has been/should be studied.)

It should be mentioned that one arrives at identically the same holonomy groupoid on A by performing a certain topological-groupoid theoretic construction on the isonomy groupoid $X_B^{-1}X$, namely "taking the β-identity component" (cf. [8] p. 45 for this construction, or rather, the analogous α-version.) I acknowledge an oral private communication from Pradines in 1988, where (if I remember correctly) he mentioned to me this relationship between isonomy and holonomy.

We may summarize our description of isonomy and holonomy groupoid in the following (with no mention of pregroupoids): an *isonomy element* or *isonomy fraction* $g^{-1}f$ is given by a pair of distinguished germs with same codomain $\in \mathbf{R}^q$; two such fractions $g^{-1}f$ and $h^{-1}k$ are equal if there is a germ λ of a diffeormorphism of $\mathbf{R}^q$ with $\lambda \circ g = h$ and $\lambda \circ f = k$. An isonomy fraction $g^{-1}f$ is a *holonomy fraction* if f and g belong to the same connected component of the fibre $\beta^{-1}(b)$ where $b \in \mathbf{R}^q$ is the (common) codomain of the germs f and g. (Then also the two fractions can be connected by a path in this fibre; this path is lift of a leafwise path in A.)

2 Topos theory of localic pregroupoids

Our interest are smooth groupoids and pregroupoids, whose domain- and codomain maps are surjective submersions, but as a tool, it is convenient to consider first localic groupoids and pregroupoids whose domain- and codomain maps are open surjections. Call for brevity such pregroupoids *good*. To fix notation, we consider $A \leftarrow X \rightarrow B$, with the two structural maps denoted α and β, respectively.

Proposition 2 *If X is a good localic pregroupoid, then $X^{-1}X$ and XX^{-1} are Morita equivalent (meaning that the classifying toposes $\mathbf{B}(X^{-1}X)$ and $\mathbf{B}(XX^{-1})$ are equivalent.*

This was essentially proved in [5] , Theorem 3.1 (combined with [4] Prop. 5.1). It can also be deduced from the proof of Proposition 4 below. One upshot of the proof in [5] is that it involves a symmetric description of a topos equivalent to as well $\mathbf{B}(X^{-1}X)$ as $\mathbf{B}(XX^{-1})$; this topos we denote $\mathbf{B}(X)$. Its objects are triples $(E \rightarrow A, F \rightarrow B, \sigma)$, where the two displayed maps are etale, and where σ is a certain map which we for simplicity describe in point-set terms: σ is a law which to each $x \in X$ associates an isomorphism from the a-fibre of E to the b-fibre of F (where $a = \alpha(x), b = \beta(x)$, and compatible with the algebraic structure (ternary operation $zx^{-1}y$) of X in an evident way (see [5]).

Proposition 3 *If $A \to \mathcal{D}$ and $B \to \mathcal{D}$ are open surjections from locales to a topos $\mathcal{D}$, then their bi-pull-back X is a locale and carries a canonical good pregroupoid structure on A B, and $\mathbf{B}(X^{-1}X) \simeq \mathcal{D} \simeq \mathbf{B}(XX^{-1})$, canonically.*

This is entirely analogous to the Proposition that the bi- pull-back of $A \to \mathcal{D}$ with itself is a locale (which defines the locale of arrows of a good localic groupoid with A for its locale of objects). (We follow Moerdijk in not distinguishing notationally between a locale A and its topos of sheaves.)

A localic pregroupoid that arises this way, we call, in analogy with the Moerdijk notion [10] for localic groupoids, *etale complete.*

Proposition 4 *If one edge groupoid of a good localic pregroupoid is etale complete, then so is the other; and also, the pregroupoid is itself etale complete.*

Proof. It can be proved as part of the "finite limits and descent" theory of pregroupoids that each of the edge groupoids are equivalent to the "diagonal" groupoid of the pregroupoid (cf. [5] Note p. 199; this is really the "Butterfly" diagram of Pradines, cf. [12] p. 536). But etale completeness is preserved under equivalence, by Proposition 3.2 in [7] . This proves the first assertion. The second assertion seems to require some diagram chasing: we argue with "generalized elements" of the locales in question, which notationally looks like we are working with point set topology. With notation as above, let $a \in A$ and $b \in B$ be given together with an isomorphism $d : p(a) \to \pi(b)$, where p and π are the augmentation maps from A and B, respectively, to $\mathbf{B}(X)$. Pick $z \in X$ with $\alpha(z) = a$. Write b' for $\beta(z) \in B$. For any object $(E, F, \sigma) \in \mathbf{B}(X)$, we have the composite isomorphism

$$F_b \overset{d^{-1}}{\to} E_a \to F_{b'},$$

the second arrow being induced via σ from z; and natural in $(E, F, \sigma) \in \mathbf{B}(X)$. By etale-completeness of XX^{-1}, there is a unique $g \in XX^{-1}(b, b')$ defining this composite. By the pull-back diagram defining the edge groupoid, there is a unique $x \in X$ with $zx^{-1} = g$, and this x will have $\alpha(x) = a$, $\beta(x) = b$ and define the given d. We omit the details.

Proposition 5 *A good localic pregroupoid $A \leftarrow X \rightarrow B$, with $X \rightarrow A$ etale, is etale-complete; and both of its edge groupoids $X^{-1}X$ and XX^{-1} are etale-complete.*

Proof. The edge groupoid $XX^{-1} \rightrightarrows B$ is etale since $X \rightarrow A$ is, so is etale complete, by Proposition 3.1 og [7] . The rest now follows from Proposition 4.

Proposition 6 *Consider classifying morphism $c_{\mathcal{F}} : A \rightarrow \mathbf{B}\Gamma^q$ of the foliation $\mathcal{F}$. Then its c.l.c./slice factorization may be identified with*

$$A \rightarrow \mathbf{B}(X_C^{-1}X) \rightarrow \mathbf{B}(X_B^{-1}X),$$

and the kernel pair groupoid of the c.l.c. factor is the holonomy groupoid $\mathbf{B}(X_C^{-1}X)$.

(The reader may recall that Moerdijk in [11] *defined* the holonomy groupoid, in terms of $c_{\mathcal{F}}$, as this kernel pair. So the Proposition says that the two definitions agree.)

Proof. Consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & C & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & \mathbf{B}(X_C^{-1}X) & \longrightarrow & \mathbf{B}(\Gamma^q) = \mathbf{B}(X_B^{-1}X) \end{array} \quad ;$$

all the arrows are immediate, the total lower arrow is the classifying geometric morphism $c_{\mathcal{F}}$. The total diagram, as well as the left hand one, are (bi-)pullbacks, since $A \leftarrow X \rightarrow B$ and $A \leftarrow X \rightarrow C$ are etale complete pregroupoids, by Proposition 5. (There is an invertible 2-cell in both of the diagrams, which is really the σ entering in the description of the topos $\mathbf{B}(X)$ above). So the right hand square is likewise a (bi-) pull-back, by a familiar descent property of (bi-) pull-backs (see e.g. [6], where it is called "co-pasting" Lemma). Since the top line by construction is the c.l.c./slice factorization of $X \rightarrow B$, and such factorization is reflected by pull-backs along open surjections (in particular along a surjective slice like $B \rightarrow \mathbf{B}(\Gamma^q)$), we conclude that the lower line in the diagram is a c.l.c/slice factorization of $c_{\mathcal{F}}$. And finally, from etale completeness of the groupoid XX_C^{-1} (which again comes from Proposition 5, we conclude that this groupoid is the kernel pair groupoid as claimed.

3 A comparison

Just for contrast, the pregroupoid X for the foliation $\mathcal{F}$ on the manifold A should be contrasted with the following pregroupoid $A \leftarrow Y \rightarrow B = \mathbf{R}^q$; Y consists of *jets* of distinguished submersions. Say 1-jets, to be specific. Then Y carries the structure of a manifold, of dimension $\dim(A) + q^2$, and receives a pregroupoid homomorphism from the previous X, namely "take jets". The map $Y \rightarrow A$ is no longer etale, and the pregroupod is far from being etale complete; in fact, there is a trivializing continuous functor $B \times B \rightarrow YY^{-1}$ (namely: to (b, c) associate the jet of the translation from b to c (utilizing that B is a vector space; on the level of germs, this construction is not continuous). This trivialization means that the pregroupoid is in a certain sense, which one can easily make precise, equivalent to an ordinary principal bundle over A (with structure group $GL(q)$).

The pregroupoid Y carries a canonical "partial connection", the Bott connection, which can be described very easily in synthetic terms, whereas X carries exactly one, trivial, connection, since $X \rightarrow A$ is etale.

4 Appendix

Recall that a geometric morphism $\beta : \mathcal{X} \rightarrow \mathcal{B}$ is called *locally connected* or *molecular* if $\beta^* : \mathcal{B} \rightarrow \mathcal{X}$ has a left adjoint $\beta_! : \mathcal{X} \rightarrow \mathcal{B}$ compatible with slicing, meaning that the canonical map

$$\beta_!(E \times \beta^* S) \rightarrow (\beta_! E) \times S$$

is an isomorphism, for all $E \in \mathcal{X}$ and $S \in \mathcal{B}$. We write 'l.c.' for 'locally connected', and 'c.l.c.' for 'connected and locally connected' (meaning that the inverse image functor is furthermore full and faithful). From [3], Proposition 4.6 (or rather from its proof), we quote most of

Proposition 7 *Any l.c. $\beta : \mathcal{X} \rightarrow \mathcal{B}$ factors, essentially uniquely, into a c.l.c. followed by a slice; the slicing object is $\beta_!(1)$. The factorization is stable under pull-back along any geometric morphism $\mathcal{P} \rightarrow \mathcal{B}$, and is reflected by pulling back along an open surjection.*

Proof. For the existence of the factorization, we refer to loc. cit. (and it is in fact very easy: the object over which we slice is just $\beta_!(1)$). The pull-back stability follows because both the property of being a slice, and the property of being c.l.c. is stable under pull-back. Also, the property of being c.l.c. is reflected by pull-backs along open surjections (cf. [9]); the property of being a slice is not in general reflected by such pull-backs. But given a l.c. geometric $\delta : \mathcal{A} \to \mathcal{B}$, factored into a c.l.c. followed by some other geometric morphism $\mathcal{A} \to \mathcal{C} \to \mathcal{B}$; let $\mathcal{A} \to \mathcal{C}' \to \mathcal{B}$ be the c.l.c. / slice factorization. Since connected geometric morphisms are orthogonal to slices (cf [3], Lemma 4.4), there is a comparison $\mathcal{C} \to \mathcal{C}'$ between the two factorizations. If now the first, given, factorization pulls back along an open surjection $\mathcal{D} \to \mathcal{B}$ to the c.l.c./slice factorization, the comparison $\mathcal{C} \to \mathcal{C}'$ pulls back to an equivalence, which implies (cf. Lemma 1 in [6]) that it was already itself an equivalence.

If $\mathcal{B}$ is of the form $sh(B)$ for some locale, respectively sober space B, then so is of course the slice $\mathcal{B}/\beta_!1$. So in particular, the factorization restricts to a factorization for locales, respectively for sober spaces; both stable under pull-back along arbitrary locale- respectively continuous map.

If X is a sober space, then $sh(X) \to sh(1) = \underline{\text{Sets}}$ is connected, respectively locally connected, if X is connected, respectively locally connected in the sense of general topology (locally connected here meaning that X has a basis consisting of connected open sets). If X is a locally connected sober space, then the middle space in the factorization $X \to 1$ is the (discrete) set of connected components of X.

Therefore, by pull-back stability of c.l.c/slice factorization, the fibres of $X \to C$, for the factorization $X \to C \to B$ of a l.c. map $X \to B$, are the connected components of the fibres of $X \to B$.

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