

Fractional Exponent Functors and Categories of Differential Equations

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1 Introduction

This paper grew out of a question of Lawvere, who asked whether categories of ordinary (1st or 2nd) order differential equations may be seen as toposes. He also gave some indications how “fractional exponents” may be used to answer this question, [25].

It is more or less classical how first order differential equations (1ODEs) are organized into a category. A 1ODE on a manifold M is the same thing as a vector field $X : M \rightarrow T(M)$, and a *morphism* of vector fields $(M_1, X_1) \rightarrow (M_2, X_2)$ is a map $f : M_1 \rightarrow M_2$ such that the square

$$\begin{array}{ccc} T(M_1) & \xrightarrow{Tf} & T(M_2) \\ \uparrow X_1 & & \uparrow X_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes. A (global) *solution* of the differential equation (M, X) , or a *flow line* of a vector field X , is a morphism from $(\mathbf{R}, \frac{\partial}{\partial x})$ to (M, X) .

Similarly, a second order differential equation (2ODE) on M is usually construed as a vector field on TM ,

$$\xi : TM \rightarrow TTM, \tag{1}$$

with certain properties, and given two such, $(M_1, \xi_1), (M_2, \xi_2)$, there is a natural notion of a morphism, namely a map $f : M_1 \rightarrow M_2$ making the

appropriate square

$$\begin{array}{ccc}
 TTM_1 & \xrightarrow{TTf} & TTM_2 \\
 \uparrow \xi_1 & & \uparrow \xi_2 \\
 TM_1 & \xrightarrow{Tf} & TM_2
 \end{array}$$

commutative.

Unlike solutions for 1ODEs, solutions for 2ODEs are not in general homomorphisms from \mathbf{R} , unless the 2ODE is a *spray*, see below.

The question of the category theoretic properties of the categories 1ODE and 2ODE thus defined depend on the nature of the underlying category of ‘spaces’ or ‘manifolds’. The context in which Lawvere posed the question was that of Synthetic Differential Geometry (SDG) where one has a *topos* \mathbf{E} of ‘spaces’, which comprise many more objects than the usual manifolds, for instance, it contains ‘infinitesimal’ objects D, D_2 etc., which classify 1-, resp. 2-jets of maps from \mathbf{R} .

In this context, the tangent bundle TM becomes the exponential object M^D . And it so happens in many of the toposes of SDG that the functor $(-)^D : \mathbf{E} \rightarrow \mathbf{E}$ not only has a left adjoint $- \times D$ but also has a right adjoint, the ‘fractional exponent’ $(-)^{1/D}$ (Lawvere’s terminology). (Objects D with this property occurred early in the history of SDG, consider [18], and they have been called *atoms* [16], *tiny objects* [30], or *amazing* [24].) Because of the extra right adjoint, the data of a 2ODE

$$M^D \rightarrow (M^D)^D \cong M^{D \times D} \tag{2}$$

may be construed as a map from M itself,

$$M \rightarrow (M^{D \times D})^{1/D}.$$

Lawvere suggested that, utilizing these fractional exponent functors, it should be possible to prove that the category of 2ODEs in \mathbf{E} is a topos (see [25]); one aim of the present article is to carry out this suggestion of Lawvere’s, and prove that the 2ODEs over \mathbf{E} do indeed form a topos. We study some of its properties.

The similar question for 1ODEs is of a more standard character, since it does not involve fractional exponents. In fact, a 1ODE (vector field) on M , $M \rightarrow M^D$, may equally be construed as a map $M \times D \rightarrow M$ with certain properties, i.e. by an *action* of the object D , and it is more or less well known that a category of such actions form a topos (related to the topos of actions-by-the-free-monoid-on- D).

The main thrust of the paper is category-theoretic: aspects of fractional exponent functors in general, and left exact endofunctors on a topos in particular, and our results in this direction may well have an independent interest. We also get the conclusion envisaged by Lawvere: that the toposes of (1ODEs and) 2ODEs resemble the topos of W -actions, for a monoid W in \mathbf{E} , namely, they “receive an essential geometric surjection p from \mathbf{E} ”, whose inverse image functor p^* is simply the forgetful functor.

An aspect, not subsumed under this topos-theoretic fact, is that 2ODEs, and related categories, are in fact canonically *enriched* in \mathbf{E} , in the sense of [8], [12]. We arrive at this enrichment by studying possible \mathbf{E} -enrichments of fractional exponent functors, in general. The main result in this direction is that there is a bijection between enrichment of a fractional exponent functor $(-)^{1/A}$ and points $1 \rightarrow A$ of the object A ; in particular, since D has a unique point (zero), $(-)^{1/D}$ carries a unique enrichment over \mathbf{E} .

Besides the category theory, we shall pursue the relationship between the notion of 2ODE and the notion of *force* of a 2ODE, relative to a given spray; this was part of Lawvere’s motivation for considering the category of 2ODEs. This will be a piece of pure SDG.

We should also mention that in so far as the notion of 2ODE goes, there is another viewpoint than the one expressed in (1), namely, that a 2ODE ξ is a map

$$M^D \rightarrow M^{D^2} \tag{3}$$

which ‘prolongs’ 1-jets into 2-jets. For sprays, this viewpoint is classical in SDG, see [7] or [22]; the general case has also been considered in [29] under the name “L-semispray”. For ‘infinitesimally linear’ (=‘microlinear’) objects M , the two notions of 2ODEs are equivalent, and we shall mainly work with (3) (which we accordingly call “version 1”; comparison with “version 2”, i.e. (2) is given in Section 7 and 10.

2 Generalities on Enrichment

We recall some notions from enriched category theory, cf. [8] or [12]. Recall that a cartesian closed category \mathbf{E} is enriched in itself (i.e. is made into an \mathbf{E} -category) by means of Y^X as the object of maps from X to Y . Then an \mathbf{E} -enrichment of an endofunctor $G : \mathbf{E} \rightarrow \mathbf{E}$ consists of a family of maps

$$G_{X,Y} : Y^X \rightarrow G(Y)^{G(X)},$$

natural in X and Y , and which satisfies two equational conditions expressing the idea that G takes identity maps to identity maps, and preserves composition. Fixing one of the variables X or Y in the exponential functor Y^X gives an endofunctor canonically enriched in \mathbf{E} .

Recall from [14] Theorem 1.3 (or [10]) that an \mathbf{E} -enrichment ("strength") of an endofunctor $G : \mathbf{E} \rightarrow \mathbf{E}$ may be encoded equivalently in the form of a "tensorial strength", meaning a family of maps

$$t_{X,Y} : X \times G(Y) \rightarrow G(X \times Y),$$

natural in X and Y , and satisfying two equational conditions,

$$t_{1,Y} = 1_{GY} : 1 \times GY \rightarrow G(1 \times Y) \quad (4)$$

$$t_{U,V \times Y} \circ (U \times t_{V,Y}) = t_{U \times V, Y} : U \times V \times GY \rightarrow G(U \times V \times Y), \quad (5)$$

respectively, for all U, V, Y , (under the evident identifications like $1 \times GY = GY$ etc.).

We also recall that there is a notion for a natural transformation $\tau_X : G_1(X) \rightarrow G_2(X)$ between two \mathbf{E} -functors to be \mathbf{E} -*natural*, or strongly natural, see [8] 1.10; in terms of the tensorial form of enrichments (for endofunctors G_1 and G_2 on \mathbf{E}), this may be expressed simply as commutativity of all squares of form

$$\begin{array}{ccc} X \times G_1 Y & \xrightarrow{t_{X,Y}^{(1)}} & G_1(X \times Y) \\ \downarrow 1 \times \tau_Y & & \downarrow \tau_{X \times Y} \\ X \times G_2 Y & \xrightarrow[t_{X,Y}^{(2)}]{} & G_2(X \times Y) \end{array} \quad (6)$$

where $t^{(i)}$ denotes the enrichment of G_i ($i = 1, 2$); cf. [14].

Recall that any endofunctor of form $(-)^A$ carries a canonical \mathbf{E} -enrichment (its monoidal form is given below).

Proposition 1 *Let $A \in \mathbf{E}$ be an object, and let $\phi_X : X^A \rightarrow X$ be natural in X . Then in order that ϕ be \mathbf{E} -natural, it is necessary and sufficient that, for all X , the composite*

$$X \xrightarrow{\Delta_X} X^A \xrightarrow{\phi_X} X$$

be the identity map on X , where $\Delta_X : X \rightarrow X^A$ denotes the ‘diagonal’-map, exponential adjoint of the projection $X \times A \rightarrow X$.

Proof. The commutative square corresponding to (6) reduces to the outer square in

$$\begin{array}{ccc} X \times Y^A & \xrightarrow{\Delta_X \times 1} & X^A \times Y^A \cong (X \times Y)^A \\ \downarrow 1 \times \phi_Y & & \searrow \phi_X \times \phi_Y \\ X \times Y & \xrightarrow{1_{X \times Y}} & X \times Y \end{array} \quad \begin{array}{c} \downarrow \phi_{X \times Y} \\ X \times Y \end{array}$$

where the upper map is the enrichment in monoidal form of $(-)^A$. Using the naturality of ϕ with respect to the two projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$, it is easy to see that under the identification $X^A \times Y^A \cong (X \times Y)^A$ $\phi_{X \times Y}$ becomes $\phi_X \times \phi_Y$, and then it is clear that $\phi_X \circ \Delta_X = 1_X$ implies commutativity of the square. The converse implication follows by taking $Y = 1$.

Remark. If $\mathbf{E} = \mathbf{Sets}$, all functors and transformations are \mathbf{E} -enriched. But for other toposes \mathbf{E} , there may exist A and natural transformations $\phi_X : X^A \rightarrow X$ which are not \mathbf{E} -natural; even for the case $A = 1$. Take e.g. $\mathbf{E} = \mathbf{Z}_2\text{-Sets}$, i.e. the topos of sets-with-an-involution, and let $\phi_X : X \rightarrow X$ be the involution on X .

Proposition 2 *Let $F \dashv G$ be endofunctors on a cartesian closed category, and assume that F preserves finite products. Given a natural $\phi_X : F(X) \rightarrow X$, then the family of maps $\tau_{X,Y}$ given by*

$$F(X \times G(Y)) \cong F(X) \times F(G(Y)) \xrightarrow{\phi_X \times \epsilon_Y} X \times Y$$

(where ϵ is the counit of the adjunction $F \dashv G$) gives by transposition along $F \dashv G$ a family of maps

$$X \times G(Y) \xrightarrow{t_{X,Y}} G(X \times Y)$$

which is an enrichment (in tensorial form) of the functor G .

Proof. Given Y , the transpose of $t_{1,Y}$ is, by construction, $\phi_1 \times \epsilon_Y$, which under identifications of type $1 \times Y = Y$ is just ϵ_Y , the transpose of the identity map on GY , proving (4). Given U, V, Y , then the right hand side in (5) has for its transpose, under identifications of type $FX \times FY = F(X \times Y)$, the map $\phi_U \times \phi_V \times \epsilon_Y$, whereas the left hand side has transpose $U \times V \times \epsilon_Y \circ (\phi_{U \times V} \times FGY)$. But just by naturality of ϕ with respect to the two projections from $U \times V$, we conclude that $\phi_{U \times V} = \phi_U \times \phi_V$, and the required commutativity is then immediate. The Proposition is proved.

Remark. We may supplement the Proposition with a statement about natural transformations $f : F_1 \rightarrow F_2$; if such an f commutes with the ‘‘augmentations’’ $\phi_i : F_i(X) \rightarrow X$, then the mate of f , $G_2 \rightarrow G_1$, becomes a strongly natural transformation, with respect to the enrichments obtained on the G_i ’s by virtue of the Proposition. In particular, the mate of ϕ itself, $X \rightarrow G(X)$, is strongly natural in X .

For any atom A , there is a canonical natural transformation $\pi_X : X^{1/A} \rightarrow X$, namely the composite

$$X^{1/A} \xrightarrow{\Delta} (X^{1/A})^A \xrightarrow{\epsilon_X} X,$$

where ϵ is the counit for the adjunction $(-)^A \dashv (-)^{1/A}$.

Theorem 1 *Let A be an atom in \mathbf{E} . There is a bijective correspondence between the set of those \mathbf{E} -enrichments of the endofunctor $(-)^{1/A} : \mathbf{E} \rightarrow \mathbf{E}$, which make the transformation $\pi : (-)^{1/A} \rightarrow id_{\mathbf{E}}$ \mathbf{E} -natural, and the set of points $1 \rightarrow A$.*

Proof. Assume given a tensorial enrichment of the endofunctor $(-)^{1/A}$,

$$X \times (Y^{1/A}) \xrightarrow{t_{X,Y}} (X \times Y)^{1/A};$$

its transpose under the adjointness $(-)^A \dashv (-)^{1/A}$ consists then in maps

$$X^A \times (Y^{1/A})^A \xrightarrow{\tau_{X,Y}} X \times Y,$$

natural in X and Y . Using this naturality with respect to the maps $X \rightarrow 1$ and $Y \rightarrow 1$, it is easy to see that $\tau_{X,Y}$ must be of form $\phi_X \times \psi_Y$ with $\phi_X : X^A \rightarrow X$ natural in X and $\psi_Y : (Y^{1/A})^A \rightarrow Y$ natural in Y . From (4) it follows that ψ_Y must in fact be the counit ϵ_Y for the adjointness $(-)^A \dashv (-)^{1/A}$. On the other hand, for any natural family $\phi_X : X^A \rightarrow X$, the transposes $t_{X,Y}$ of the maps $\tau_{X,Y} := \phi_X \times \epsilon_Y$ will in fact provide an enrichment for $(-)^{1/A}$, without any assumptions on ϕ except naturality; this follows by taking $F = (-)^A$, $G = (-)^{1/A}$ in Proposition 2.

From the analysis made it follows that there is a bijective correspondence between enrichments $t_{X,Y} : X \times (Y^{1/A}) \rightarrow (X \times Y)^{1/A}$ of the endofunctor $(-)^{1/A}$, and natural transformations $\phi_X : X^A \rightarrow X$.

Now compatibility of the enrichment t with π is easily seen, by transposition, to be equivalent to the normalization condition $\phi_X \circ \Delta_X = id_X$.

But by Proposition 1, this condition is equivalent to \mathbf{E} -naturality of $\phi_X : X^A \rightarrow X$. Finally, we invoke the enriched Yoneda Lemma, in the form of [12] 1.9 (therein called the *weak* Yoneda, since after all it talks about a bijection between two *sets!*). According to it, the set of \mathbf{E} -natural transformations $(-)^A \rightarrow (-)^B$ is in bijective correspondence with the set of maps $B \rightarrow A$. Now take $B = 1$. The Theorem is proved.

Remark. On a cartesian closed category, any endofunctor $- \times A$ is the functor part of a comonad, with counit and comultiplication being $- \times (A \rightarrow 1)$ and $- \times \Delta_A$, respectively. It follows by mating that $(-)^A$ carries a canonical structure of monad. If A is an atom, then, again by mating, $(-)^{1/A}$ carries a canonical structure of comonad. The counit of this comonad structure is the π considered in Theorem 1. So if $(-)^{1/A}$ is supplied with a strength, by virtue of a point of A , as in the Theorem, π is strongly natural. It is natural to ask whether also the comultiplication of the comonad is strongly natural, (so that the comonad becomes a strong one). The answer is yes. This follows in essence by the Remark after Proposition 2; for, a point o of

A induces an augmentation of $((-)^A)^A$ (just evaluate twice at o), with which the multiplication of the monad $(-)^A$ is compatible. (Note that the monad $(-)^A$ carries a canonical strength, but that strength does not transfer to a strength on $(-)^{1/A}$, since the adjointness $(-)^A \dashv (-)^{1/A}$ is *not* strong, see below.)

3 Adjoint Monads

An adjoint monad on \mathbf{E} , is an adjoint monad/comonad pair, as in [9], [26] V.8, or [3] 3.7 (who call such things *adjoint triple*¹.) Lawvere [25] calls such things a *Galilean Monoid* in \mathbf{E} ; he argues that in some respects they are like (monads defined by) monoid objects. Note that the monad or comonad, or the adjunction, need not be \mathbf{E} -enriched. For instance, to say that the adjunction $(-)^A \dashv (-)^{1/A}$ is enriched implies that there are natural isomorphisms

$$Y^{(X^A)} \cong (Y^{1/A})^X$$

(whose global sections give the usual hom-set bijection for adjointness), and taking $X = 1$, one sees that, naturally in Y , $Y \cong Y^{1/A}$; but if $(-)^{1/A}$ is isomorphic to the identity functor, then so are, in turn, its adjoint $(-)^A$ and the adjoint $A \times -$ of this last one. It follows that $A = 1$.

One of the early results in monad theory is the fact than any 3-chain of functors $p_! \dashv p^* \dashv p_*$ gives rise to an adjoint monad/comonad pair on the domain \mathbf{E} of $p_!$; and also that if p^* reflects isomorphisms, and the domain category \mathbf{G} of p^* is sufficiently complete, p^* makes \mathbf{G} monadic, as well as comonadic, over \mathbf{E} ; cf. [5] Theorem 2.11, (or [3] 3.7).

We shall give a proof of another result in this direction, suggested by Lawvere (possibly, it is in [5]?).

Proposition 3 *The only \mathbf{E} -enriched adjoint monads in a topos \mathbf{E} are those that come from actual monoids in \mathbf{E} .*

(To say that the adjoint monad is enriched we take to mean that the functors, as well as the transformations that make up the monad and its adjoint comonad are enriched, and also that the adjunction $T \dashv G$ itself is enriched, $G(Y)^X \cong Y^{T(X)}$.)

¹We require, unlike [3], that the obvious compatibility conditions between the unit of the monad, and the counit of the comonad, and similarly for the (co-)multiplication, hold.

Proof. Since the monad \underline{T} is enriched, we have from [13] that its functor part T carries a canonical structure of monoidal functor, in particular, it takes monoids to monoids. Since the terminal object 1 carries a monoid structure trivially, we get a monoid structure m on $T(1)$. Due to the trivial nature of 1 , this structure may be described a little simpler than in [13], namely as $m =$

$$T1 \times T1 \xrightarrow{t_{T1,1}} T(T1 \times 1) \cong TT1 \xrightarrow{\mu_1} T1,$$

where μ is the multiplication of the monad. The unit η_1 immediately furnishes the unit for the monoid $T1$.

We now prove that the given monad \underline{T} is isomorphic to the monad $X \mapsto X \times T1$. First, in so far as the functor part is concerned: since the \mathbf{E} -functor T has an \mathbf{E} -adjoint G , it follows that it preserves tensors (internal copowers), (see 4.1 in [11] for the dual statement), which here means that the maps

$$t_{X,Y} : X \times T(Y) \rightarrow T(X \times Y)$$

are isomorphisms. In particular $X \times T1 \cong T(X)$ via $t_{X,1}$, meaning that the functor part of the monad \mathbf{T} agrees with $-\times T1$. We have to see that the μ_X of the monad agrees with $X \times m : X \times T1 \times T1$ (and correspondingly, but easier, for η). This we get by contemplating the following diagram, which is commutative because of the assumption that $\mu : T \circ T \rightarrow T$ is \mathbf{E} -natural:

$$\begin{array}{ccc} X \times T^2 1 & \xrightarrow{\bar{t}_{X,1}} & T^2 X \\ \downarrow X \times \mu_1 & & \downarrow \mu_X \\ X \times T1 & \xrightarrow{t_{X,1}} & TX, \end{array}$$

where \bar{t} denotes the monoidal strength of T^2 , derived from the monoidal strength t of T ,

$$\bar{t}_{X,Y} = T(t_{X,Y}) \circ t_{X,TY}.$$

Since m is derived from μ_1 , the diagram does relate $X \times m$ with μ_X , after suitable identifications (via instances of t) of $X \times T1 \times T1$ with $X \times T^2 1$.

A Corollary is that the only adjoint monads in the category of Sets are the actual monoids. (This was first observed in [5], Introduction.) For, all functors are vacuously Set-enriched.

4 Topos theoretic aspects

Recall that if $G : \mathbf{E} \rightarrow \mathbf{E}$ is an endofunctor on a category, then the category $\text{Coalg}(G)$ of coalgebras for it is the category of pairs (X, ξ) where $X \in \mathbf{E}$ and $\xi : X \rightarrow G(X)$. (Coalgebras for a comonad $\underline{G} = (G, \epsilon, \delta)$ will be discussed later in this Section; the category of such coalgebras will be denoted $\text{COALG}(\underline{G})$)

Proposition 4 *Let \mathbf{E} be a Grothendieck topos, and G a left exact accessible endofunctor on it. Then the category $\text{Coalg}(G)$ of coalgebras for G is a Grothendieck topos; the forgetful functor to \mathbf{E} is the inverse image functor of a surjective geometric morphism.*

Proof. The comma category $\mathbf{E} \downarrow G$ is a Grothendieck topos, according to SGA 4, [2] IV.9.5 Theorem 4; they call it the topos obtained by *glueing* (recollement) along G . The forgetful functor $\mathbf{E} \downarrow G \rightarrow \mathbf{E} \times \mathbf{E}$ is the inverse image functor of a geometric morphism of Grothendieck toposes; this is likewise (implicit) in loc. cit. If we form the (strict) pull-back of categories of this forgetful functor along the diagonal $\mathbf{E} \rightarrow \mathbf{E} \times \mathbf{E}$, we obtain the category of G -coalgebras. But the functors in this pull-back diagram are all inverse image functors of geometric morphisms; this follows from [27], according to which colimits in the category of Grothendieck toposes and geometric morphisms is formed by forming limits of the inverse-image functors.

Proposition 5 *Let \mathbf{E} be a Grothendieck topos, and G a endofunctor on it which admits a left adjoint T . Then the category of coalgebras for G (which is \cong to the category of T -algebras) is a Grothendieck topos; the forgetful functor to \mathbf{E} is the inverse image functor of an essential and surjective geometric morphism.*

Proof. Any (left or right) adjoint functor between Grothendieck toposes is accessible, see [2] Proposition I.9.5. So by Proposition 4, the category of coalgebras for G is a Grothendieck topos. Using that G preserves limits, it is easy to see that the forgetful functor (creates and) preserves limits. Also the forgetful functor (creates and) preserves colimits, hence is accessible. But an accessible limit preserving functor between Grothendieck toposes has a left adjoint, by the Adjoint Functor Theorem (in the form of [1] 1.66 (p. 52), say).

(A similar argument proves that the glueing topos for a glueing functor $G : \mathbf{E} \rightarrow \mathbf{F}$ which is a right adjoint (hence accessible, cf. [2] Prop. I.9.5) receives an essential geometric morphism from the product category $\mathbf{E} \times \mathbf{F}$ (which is the coproduct in the category of toposes), with inverse image functor the obvious forgetful functor).

For a comonad $\underline{G} = (G, \epsilon, \delta)$, on \mathbf{E} , it is classical, cf. [20], or [26] V.8 Theorem 4, that the category $COALG(\underline{G})$ of coalgebras for it (coalgebras in the sense of monad/comonad theory) is an elementary topos. It is probably true, but we haven't been able to find a proof or reference, that if G is accessible, this elementary topos is even a Grothendieck topos. When G has a left adjoint T , and so is part of an adjoint monad, this conclusion is easily proved. For, the forgetful functor $COALG(\underline{G}) \rightarrow \mathbf{E}$ is monadic as well as comonadic, as was observed by Eilenberg and Moore, see [26] V.8 Theorem 2; so it has all the completeness and exactness properties required in the Giraud characterization of Grothendieck toposes. So we only have to argue for the existence of a set of generators. But \mathbf{E} has a set $(X_i)_{i \in I}$ of generators; if F denotes the left adjoint of the forgetful functor $COALG(\underline{G}) \rightarrow \mathbf{E}$, it is clear that the $F(X_i)$ is a set of generators for $COALG(\underline{G})$, which thus is a Grothendieck topos. For later reference we record this information in

Proposition 6 *Let \mathbf{E} be a Grothendieck topos, and let $\underline{G} = (G, \epsilon, \delta)$ be a comonad on \mathbf{E} such that the endofunctor G has a left adjoint. Then the category $COALG(\underline{G})$ of coalgebras for G is a Grothendieck topos, and the forgetful functor to \mathbf{E} is the inverse image functor of a surjective geometric morphism.*

We need one more purely category theoretic result: let $F, G : \mathbf{A} \rightarrow \mathbf{B}$ be functors, and $\phi, \psi : F \rightarrow G$ natural transformations, so that the equifier $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ of ϕ and ψ (= the full subcategory of \mathbf{A} given by those $X \in \mathbf{A}$ such that $\phi_X = \psi_X$) makes sense. Then

Proposition 7 *If \mathbf{A} has and F preserves a certain class of colimits, then $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ is closed under this class of colimits; if \mathbf{A} has and G preserves a certain class of limits, then $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ is closed under this class of limits. And if G preserves monics, then $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ is closed under subobjects.*

Proof. For the first assertion, we want to see that ϕ and ψ agree for an object of form $\lim_{\rightarrow} X_i$, provided they do so for the X_i 's, so we want to prove

that two arrows $F(\lim_{\rightarrow} X_i) \rightarrow G(\lim_{\rightarrow} X_i)$, are equal. But the domain here is by assumption a colimit of $F(X_i)$'s on which ϕ and ψ agree, and the rest then follows by naturality. The two other assertions are proved in a similar way.

Theorem 2 *With the notation of the previous Proposition, if \mathbf{A} is a Grothendieck topos, and F is cocontinuous and G left exact, then the equifier $\mathbf{E}(\phi, \psi)$ is a Grothendieck topos, and the inclusion $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ is the inverse image functor of a surjective geometric morphism. If G preserves all limits, this geometric functor is even essential.*

Proof. By the previous Proposition, the equifier subcategory \mathbf{E} is closed in \mathbf{A} under colimits, finite limits, and subobjects. The two first of these properties guarantee that \mathbf{E} has the same exactness properties (involving these kinds of colimits/limits) as \mathbf{A} does; so \mathbf{E} satisfies the exactness properties a), b), c) of the Giraud characterization of Grothendieck toposes in [2] IV.1.1.2. To prove the last property d) (existence of a small generating family), take a small generating family K in \mathbf{A} closed under quotients. Because the family is closed under quotients, every object X in \mathbf{A} is covered by a family of *monics* with domain in K (take the *images* of the maps from objects of K to X). Since the equifier subcategory \mathbf{E} is closed under subobjects, it follows that if $X \in \mathbf{E}$, it is covered by a family of maps with their domains in the small family $K \cap \mathbf{E}$. (This proof of existence of a small generating subcategory for \mathbf{E} was pointed out to us by Ieke Moerdijk.) So \mathbf{E} is a Grothendieck topos. For the last assertion (existence of left adjoint for the inclusion), we first get from Proposition 7 that the inclusion $\mathbf{E} \subseteq \mathbf{A}$ preserves all limits. So the conclusion again follows from the Adjoint Functor Theorem, in the form quoted above.

We now consider a fixed map $\alpha : A \rightarrow E$ in a Grothendieck topos \mathbf{E} , and we assume that A is an *atom* in the sense that the endofunctor $(-)^A : \mathbf{E} \rightarrow \mathbf{E}$ has a right adjoint $(-)^{1/A}$. The composite endofunctor G given by

$$X \mapsto (X^E)^{1/A}$$

is then left exact, it even has a left adjoint. So by Proposition 5, the category of coalgebras for G is a Grothendieck topos, and the forgetful functor to \mathbf{E} is the inverse image functor p^* of an essential geometric surjection. Note that for a coalgebra (X, ξ) ,

$$\xi : X \rightarrow (X^E)^{1/A},$$

the structure map ξ corresponds by adjointness to a map

$$\xi' : X^A \rightarrow X^E.$$

We will be interested in the full subcategory consisting in those coalgebras (X, ξ) for which ξ' has the property that

$$X^\alpha \circ \xi' = \text{identity map of } X^A$$

(i.e. ξ' restricts to the identity map along α). This subcategory is easily seen to be a equifier subcategory of the kind dealt with in Proposition 2. In fact, consider the functor $q = (-)^A \circ p^* : \text{Coalg}(G) \rightarrow \mathbf{E}$ which takes (X, ξ) to X^D . We have two transformations from q to q ; the one is the identity transformation, call it ϕ , the other, ψ is the one whose component at $(X, \xi) \in \text{Coalg}(G)$ is $X^\alpha \circ \xi'$; and the equifier of these two is clearly the subcategory in question. Now since q preserves limits as well as colimits, it follows from Theorem 2 that the equifier subcategory inside the category of coalgebras is a Grothendieck topos, and that the inclusion to $\text{Coalg}(G)$, and hence the forgetful functor to \mathbf{E} , is the inverse image of an essential geometric surjection.

We then have

Theorem 3 *Let $\alpha : A \rightarrow E$ be a map in a Grothendieck topos \mathbf{E} , and assume that A is an atom. Then the category \mathbf{G} whose objects are pairs (X, ξ') with $X \in \mathbf{E}$ and $\xi' : X^A \rightarrow X^E$ with*

$$X^A \xrightarrow{\xi'} X^E \xrightarrow{X^\alpha} X^A = \text{id}_{X^A}$$

(and evident morphisms) is a Grothendieck topos, and the forgetful functor $(X, \xi') \mapsto X$ is the inverse image functor of an essential geometric surjection $\mathbf{E} \rightarrow \mathbf{G}$.

Proof. This follows from the above by noting that the category in question is equivalent to the category of (X, ξ) with $\xi : X \rightarrow (X^E)^{1/A}$ satisfying $X^\alpha \circ \xi' = \text{identity map of } X^A$, where ξ and ξ' correspond under the adjointness $(-)^A \dashv (-)^{1/A}$.

Remark. Since the inverse image functor $\mathbf{G} \rightarrow \mathbf{E}$ preserves finite products, it is a *closed* functor, so transforms \mathbf{G} -enrichment into \mathbf{E} -enrichment, in particular, transforms the \mathbf{G} -enrichment of \mathbf{G} given by its cartesian closed structure, into an enrichment of \mathbf{G} in \mathbf{E} . We do not know much about this enrichment, since we do not know much about how exponentials look like in \mathbf{G} . In particular, we don't know at present to what extent this enrichment can be derived from the one we now are going to consider.

5 Enrichment of the Coalgebra Categories

The categories which we constructed in Section 4, often can be enriched in the base topos \mathbf{E} ; in fact, a sufficient condition is that the endofunctor G carries an \mathbf{E} -enrichment, which will be the case when the atom A used for construction of G has a point, as we demonstrated in Section 2.

For the present purpose, it is better to have the \mathbf{E} -enrichment of G encoded not in tensorial form $X \times GY \rightarrow G(X \times Y)$, but rather in the classical form, as a family of maps $G_{X,Y} : [X, Y] \rightarrow [GX, GY]$ (where square brackets denote hom-objects, as in [12]). We shall write st (for “*strength of G*”) rather than $G_{X,Y}$, to save subscripts. Also, when the bifunctor $[X, Y]$ is applied to a map ξ , say, we sometimes write $[\xi, 1]$ as ξ^* and $[1, \xi]$ as ξ_* ; this is also standard mathematical usage for the contra-, respectively covariant, aspect of hom-functors.

Proposition 8 *Let $G : \mathbf{E} \rightarrow \mathbf{E}$ be a \mathbf{V} -functor, where \mathbf{V} is a symmetric monoidal closed category with equalizers. Then the category $\text{Coalg}(G)$ of G -coalgebras carries a canonical \mathbf{V} -enrichment.*

We are interested only in the case where $\mathbf{V} = \mathbf{E}$, and \mathbf{E} is a topos, hence cartesian closed, and accordingly, we shall write \times rather than \otimes for the monoidal structure. But we shall write $[X, Y]$ rather than Y^X , for typographic convenience.

Proof/construction. This is straightforward, in the spirit of [5] or [15] (in fact, we could probably read off the desired conclusion from either, - say from the proof of (2.5) in [15], by suitable dualization and exponential adjoints). But we shall be more explicit: to construct the \mathbf{V} -valued hom $[[X, Y]]$ for two G -coalgebras $X = (X, \xi)$ and $Y = (Y, \eta)$ (where $\xi : X \rightarrow GX$, $\eta : Y \rightarrow GY$), we take the equalizer object in the equalizer diagram

$$[[X, Y]] \xrightarrow{i_{X,Y}} [X, Y] \begin{array}{c} \xrightarrow{\xi^* \circ st_{X,Y}} \\ \xrightarrow{\eta_*} \end{array} [X, GY]$$

If (Z, ζ) is a third G -coalgebra, we would like to prove that the composition map

$$[nohug][Y, Z] \times [X, Y] \xrightarrow{M_{XYZ}} [X, Z]$$

restricts to a map

$$[[Y, Z]] \times [[X, Y]] \longrightarrow [[X, Z]].$$

Now consider the following two maps $[Y, Z] \times [X, Y] \rightarrow [X, GZ]$; the first is

$$[Y, Z] \times [X, Y] \xrightarrow{1 \times \eta_*} [Y, Z] \times [X, GY] \xrightarrow{st \times 1} [GY, GZ] \times [X, GY] \xrightarrow{M} [X, GZ], \quad (7)$$

the second is

$$[Y, Z] \times [X, Y] \xrightarrow{st \times 1} [GY, GZ] \times [X, Y] \xrightarrow{\eta^* \times 1} [Y, GZ] \times [X, Y] \xrightarrow{M} [X, GZ]. \quad (8)$$

It is a consequence of the extraordinary naturality [K:E] of M w.r.to η that these two maps are equal.

Now the strategy is to eliminate η in (7) in favour of ξ , using $i_{X,Y}$, and to eliminate η from (8) in favour of ζ , using $i_{Y,Z}$. More precisely, consider the map

$$[Y, Z] \times [X, Y] \xrightarrow{M} [X, Z] \xrightarrow{st} [GX, GZ] \xrightarrow{\xi^*} [X, GZ]. \quad (9)$$

We claim that the restrictions of (7) and (9) along $1 \times i_{X,Y}$ are equal. In (9), use $st \circ M = M \circ (st \times st)$, which is a general property of enrichments st of functors G (“ G commutes with composition”). Also, naturality of M w.r.to ξ gives the first equality sign in

$$\begin{aligned} \xi^* \circ M \circ (st \times st) &= M \circ (1 \times \xi^*) \circ (st \times st) \\ &= M \circ st \times 1 \circ (1 \times \xi^*) \circ (1 \times st) \end{aligned}$$

(the second equality sign by bifactoriality of \times). When restricted along $1 \times i_{X,Y}$, the factor $(1 \times \xi^*) \circ (1 \times st)$ may be replaced by $1 \times \eta_*$, so that the total expression gets replaced by (7).

Also, consider the map

$$[Y, Z] \times [X, Y] \xrightarrow{M} [X, Z] \xrightarrow{\zeta_*} [X, GZ], \quad (10)$$

or, equivalently by naturality of M w.r.to ζ ,

$$[Y, Z] \times [X, Y] \xrightarrow{\zeta_* \times 1} [Y, GZ] \times [X, Y] \xrightarrow{M} [X, GZ]. \quad (11)$$

When restricted along $i_{Y,Z} \times 1$, the composite (11) yields the same as does (8). We conclude that (7) and (8) have the same restriction along $i_{Y,Z} \times i_{X,Y}$. In formula

$$\zeta_* \circ M \circ i \times i = \xi^* \circ st \circ M \circ i \times i,$$

which is precisely the condition for $m \circ i \times i$ to factor across the equalizer $i_{X,Z}$ of ζ_* and $\xi^* \circ st$.

6 Applications to SDG

We consider in this section a Grothendieck topos \mathbf{E} which is a model of SDG, so we assume that there is given a ring object R in \mathbf{E} which satisfies the standard basic axioms of SDG. In particular, we consider the objects $D \subseteq R$, $D_2 \subseteq R$, $D(2) \subseteq R \times R$ of elements of square zero, respectively of cube zero, resp.... Theorem 3 has the following Corollary for 1ODEs and 2ODEs; we shall, in the proof recall the meaning of these notions in the synthetic context. The notion "preconnection" also mentioned in the statement of the Theorem is an ad hoc one, which will also be explained; it serves just as a stepping stone to linear as well as non-linear connections.

Theorem 4 *Let \mathbf{G} be any of the following categories:*

- (i) *Vector fields (1ODEs) on \mathbf{E} ;*
- (ii) *Second order differential equations (2ODEs) on \mathbf{E} ;*
- (iii) *Preconnections on \mathbf{E} .*

Then \mathbf{G} is a Grothendieck topos, and the forgetful functor $(M, \xi) \mapsto M$ is the inverse image of an essential geometric surjection $\mathbf{E} \rightarrow \mathbf{G}$.

Proof. (i) Recall that a vector field in \mathbf{E} (more precisely: a vector field on the object $M \in \mathbf{E}$) is a pair (M, ξ) with $M \in \mathbf{E}$ and $\xi : M \rightarrow M^D$ with

$$M \xrightarrow{\xi'} M^D \xrightarrow{M^0} M = \text{identity on } M,$$

where $0 : 1 \rightarrow D$. The result follows from Theorem 4 by letting $\alpha : A \rightarrow E$ be $0 : 1 \rightarrow D$. (This case could also be proved by construing the category of vector fields as the actions by a certain monoid in \mathbf{E} generated by D .)

(ii) Similarly, a 2ODE in \mathbf{E} is a pair (M, ξ) with $M \in \mathbf{E}$ and $\xi : M^D \rightarrow M^{D^2}$ such that the composite

$$M^D \xrightarrow{\xi} M^{D^2} \xrightarrow{M^i} M^D = \text{identity on } M^D,$$

where $i : D \rightarrow D_2$ is the canonical inclusion. We get this case from the Theorem by putting α equal to this inclusion i .

(iii) A *pre-connection* is a pair (M, ξ) with $M \in \mathbf{E}$ and $\xi : M^{D(2)} \rightarrow M^{D \times D}$ with

$$M^{D(2)} \xrightarrow{\xi} M^{D \times D} \rightarrow M^{D(2)} = \text{identity on } M^{D(2)},$$

where $i : D(2) \rightarrow D \times D$ is the canonical inclusion.

There are some further results of the same kind, which follow from Theorem 2, but apparently not from Theorem 3. Again, the differential-geometric notions entering will be recalled; and again, we present one which is ad hoc, and which we call *general force systems*. The name derives from the fact that the difference of two 2ODEs on an infinitesimally linear object can be defined as such a structure, and is to be seen as the *force* of the one relative to the other (which is usually taken to be a spray); see [25].

Theorem 5 *Let \mathbf{E} be a Grothendieck topos which is a model of SDG, and let \mathbf{G} be any of the following categories*

(iv) *Sprays in \mathbf{E}*

(v) *Linear (= affine) connexions (and also symmetric affine connexions, and also non-linear connexions) in \mathbf{E}*

(vi) *General force systems in \mathbf{E}*

(vii) *Tensors of type $(1, 1)$, and also almost complex structures, in \mathbf{E} .*

Then \mathbf{G} is a Grothendieck topos, and the forgetful functor $(X, \xi) \mapsto X$ is the inverse image of an essential geometric morphism.

Proof. (iv) Recall that a *spray* is a second order differential equation $\xi : M^D \rightarrow M^{D_2}$ which is equivariant with respect to the multiplicative actions of (R, \cdot) on M^D and M^{D_2} respectively. (Both these actions are given in turn by the multiplicative actions of (R, \cdot) on D and D_2 , respectively, thus for t in M^D or in M^{D_2} and $\lambda \in R$

$$(\lambda \cdot t)(d) := t(\lambda \cdot d),$$

for $d \in D$, respectively $d \in D_2$.) The equivariance of ξ can be expressed by the equality of two maps (defined in terms of ξ and the R -actions) with domain $R \times M^D$ and codomain M^{D_2} . Taking exponential adjoints of these two maps, the equivariance of ξ can be expressed by equality of two maps with domain M^D and codomain $M^{D_2 \times R}$. Taking, in turn, the adjoints of

these two maps with respect to the adjointness $(-)^D \dashv (-)^{1/D}$, we get that the equivariance can be expressed in terms of equality of two maps

$$M \rightarrow (M^{D_2 \times R})^{1/D},$$

both of which are constructed from ξ and hence clearly are natural in $(M, \xi) \in \mathbf{F}$, where \mathbf{F} denotes the topos of 2ODE's. They are both natural transformations from the forgetful functor

$$\mathbf{F} \rightarrow \mathbf{E} \text{ given by } (M, \xi) \mapsto M$$

to the functor

$$\mathbf{F} \rightarrow \mathbf{E} \text{ given by } (X, \xi) \mapsto (X^{D_2 \times R})^{1/D},$$

and since the former preserves colimits and the latter limits, Theorem 2 applies. We thus get that the inclusion from the equifier of these two transformations to \mathbf{F} is the inverse image of an essential geometric surjection. Composing this inclusion with the forgetful functor $\mathbf{F} \rightarrow \mathbf{E}$, which is likewise the inverse image functor of an essential geometric surjection, we get the result for sprays, as claimed.

(v) This is analogous: recall that to say that a preconnexion $\xi : M^{D(2)} \rightarrow M^{D \times D}$ is *affine*, or *linear* is to say that it is equivariant with respect to the two multiplicative actions of $R \times R$ on $M^{D(2)}$ and $M^{D \times D}$ given by

$$(\lambda_1, \lambda_2) \cdot \tau)(d_1, d_2) := \tau(\lambda_1 d_1, \lambda_2 d_2)$$

for $\tau \in M^{D(2)}$, respectively $\in M^{D \times D}$ and $(d_1, d_2) \in D(2)$, respectively $\in D \times D$. The proof is then similar to the one given for sprays, except we use $R \times R$ rather than R . - The case of symmetric linear connexions is similar:: this means linear connexions $M^{D(2)} \rightarrow M^{D \times D}$ which are equivariant with respect to the "twist" maps σ on $M^{D(2)}$ and $M^{D \times D}$ resulting from the twist maps $(d_1, d_2) \mapsto (d_2, d_1)$ on $D(2)$ and on $D \times D$: this is again an equifier subcategory of the category of linear connexions. Finally, a non-linear connection is a preconnection which is equivariant with respect to one (say, the second) action of (R, \cdot) , but not necessarily with respect to the other one (these are studied, in their relationship to 2ODEs, in [29]). The functors in question are 1) the forgetful functor from the category of connexions (resp. affine connexions) to \mathbf{E} , and 2) the functor

$$(M, \xi) \mapsto (M^{D \times D})^{1/D(2)}$$

and the natural transformations ϕ and ψ are those whose component at (M, ξ) is ξ' and $(\xi^\sigma)'$ where these are the maps

$$M \rightarrow (M^{D \times D})^{1/D(2)},$$

transposes of ξ and ξ^σ under the adjointness $(-)^{D(2)} \dashv (-)^{1/D(2)}$. (Here, $(-)^{\sigma}$ denotes "conjugation by σ " in an evident sense.)

(vi) A general force system on $M \in \mathbf{E}$ is a map $\phi : M^D \rightarrow M^D$, commuting with the projection $M^D \rightarrow M$

(vii) A *tensor* of type (1,1) on M may be construed as a force system $I : M^D \rightarrow M^D$ over M , equivariant for the multiplicative action by R on M^D ; it is an *almost complex structure* if it further satisfies $I \circ I =$ multiplication by $-1 \in R$. Again, the structure $I : M^D \rightarrow M^D$ itself gives by adjointness a coalgebra structure $I' : M \rightarrow (M^D)^{1/D}$ for the endofunctor G given by $M \mapsto (M^D)^{1/D}$, and the conditions give rise to equifier constructions giving subcategories of the topos of coalgebras; in fact the equivariance condition is dealt with as before, and the condition with minus is an equifier where the domain functor is the forgetful, the codomain is $M \mapsto (M^D)^{1/D}$, and the two transformations are the transpose of $I \circ I$ and the transpose of multiplication by $-1 : M^D \rightarrow M^D$, respectively.

Remark 1. Note that for the case of force systems, the endofunctor $M \mapsto (M^D)^{1/D}$ carries a monad structure; this structure does not enter into the description.

Remark 2. The number of examples can be vastly expanded.

7 Algebraic Comparison Functors

In the results of the previous paragraph, the various "forgetful" functors $\mathbf{G} \rightarrow \mathbf{E}$ from the toposes \mathbf{G} constructed were described as inverse image functors p^* of essential geometric surjections p from the base topos \mathbf{E} . We now change the viewpoint into a more algebraic one. Since p is an essential geometric surjection, p^* has adjoints on both sides, and is conservative (reflects isomorphisms). It is well known that these properties imply that p^* is in fact monadic and comonadic, see [5] Theorem 2.11, (or [3] 3.7 Proposition 4). This implies by a well known adjoint lifting theorem (cf. e.g. [4] II 4.5.6) that any comparison functor $q : \mathbf{G} \rightarrow \mathbf{G}'$ commuting with the underlying functors, has in fact adjoint on both sides. We shall make a couple of cases of this explicit.

Recall that the addition $+$: $R \times R \rightarrow R$ on the number line R restricts to a map $D \times D \rightarrow D_2$, likewise denoted $+$, thereby inducing a map

$$M^{D_2} \xrightarrow{M^+} M^{D \times D} \cong (M^D)^D.$$

If $\xi : M^D \rightarrow M^{D_2}$ is a 2ODE on M , the composite

$$M^D \rightarrow M^{D_2} \rightarrow (M^D)^D \tag{12}$$

is easily seen to be a vector field on M^D , in fact even more, it is a “second order differential equation version 2” on M . Denoting the category of 2ODEs (version 1) by \mathbf{G}_1 , and that of 2ODE (version 2) by \mathbf{G}_2 , we have a functor $q^* : \mathbf{G}_1 \rightarrow \mathbf{G}_2$, commuting with the underlying functors to \mathbf{E} (the underlying functor taking the displayed 2ODE of the first or second kind to the object $M \in \mathbf{E}$). By the adjoint lifting theorem quoted (applied to the identity functor on \mathbf{E}), we get that q^* has adjoints on both sides. It is also clearly conservative. Hence it may be seen as the inverse image functor of an essential geometric surjection $q : \mathbf{G}_2 \rightarrow \mathbf{G}_1$, satisfying $q \circ p_2 = p_1$.

The functor $q^* : \mathbf{G}_1 \rightarrow \mathbf{G}_2$ comes close to being an equivalence; it is well known that if M is infinitesimally linear, q^* provides a bijective correspondence between 2ODE’s on M in the sense of version 1, and 2ODEs on M in the sense of version 2. (The proof is the same as that of Proposition 5.9 in [22].)

For a second example, consider the (full and faithful) inclusion i^* from the category of sprays to the category 2ODE (version 1). Since it commutes with underlying functors to \mathbf{E} , i^* likewise has adjoints on both sides. (Of course i^* is far from being an equivalence.) It is likewise the inverse image functor of an essential geometric surjection.

For the final example, consider again $\mathbf{G}_1 = 2ODE$, the category of 2ODEs (version 1), and also the category 1ODE of 1ODE’a (vector fields). Consider the commutative square

$$\begin{array}{ccc} 2ODE & \xrightarrow{q^*} & 1ODE \\ p_1^* \downarrow & & \downarrow h^* \\ \mathbf{E} & \xrightarrow{(-)^D} & \mathbf{E}. \end{array}$$

Here, q^* is essentially the same construction as before, but now we view the composite (12) as a vector field on M^D rather than as a 2ODE (version 2) on M . Again, since the functor $(-)^D$ has adjoints on both sides, the adjoint lifting theorem implies that so does the functor on the top. As in the previous examples, this functor is the inverse image of an essential geometric surjection.

For the functor on the right, we have some idea of what the adjoints look like. In fact, since \mathbf{E} has a natural-numbers object, it is clear that one can construct the free monoid on D , modulo the equation that $0 \in D$ is to be the neutral element of the monoid; denote this monoid object by $W_0(D)$. Then clearly the set of (left) monoid-actions by $W_0(D)$ on an object M is in bijective correspondence with the set of maps $X : D \times M \rightarrow M$ with $X(0, m) = m$ for all $m \in M$; such a map is just a vector field (1ODE) on M , and in fact, the category $1ODE$ is by this process equivalent to the category of $W_0(D)$ -actions (actions in the sense of monoid theory); and here the adjoints of the forgetful functor are known: the left adjoint h_l takes M to $W_0(D) \times M$ with suitable action, the right adjoint h_* takes it to $M^{W_0(D)}$, with suitable action. In particular, $h_l(1) = W_0(D)$.

We have not been able to prove that the category $2ODE$ is not the category of actions by a suitable monoid. But for the category of sprays, this negative result is easy: we claim the left adjoint of the forgetful from sprays to \mathbf{E} preserves terminal objects, so cannot be of the form $H \times -$ for a monoid H unless H is trivial, $H = 1$.

To prove the claim, let (M, ξ) be an object equipped with a spray $\xi : M^D \rightarrow M^{D^2}$, and let $m : 1 \rightarrow M$ be a point of M . To see that the (unique) spray on 1 makes m into a homomorphism of sprays means to see that $\xi \circ m^D = m^{D^2}$, under the identifications $1 = 1^D = 1^{D^2}$. But this follows from commutativity of the square

$$\begin{array}{ccc}
 M & \xrightarrow{\Delta} & M^{D^2} \\
 \uparrow 1 & & \uparrow \xi \\
 M & \xrightarrow{\Delta} & M^D,
 \end{array}$$

(which is a consequence of the homogeneity of ξ with respect to the scalar

0). (Actually, it is probably worthwhile to consider the category (topos) of those 2ODE ξ that have this 0-homogeneity property, i.e, which make the displayed square commute.)

8 Solutions vs. Homomorphisms

We investigate here an aspect of the categories of vector fields (1ODEs), second order differential equations, and sprays, from the viewpoint of *solutions* of differential equations. To what extent may a solution of a differential equation be construed as a homomorphism from a suitable standard object ? We first describe the standard objects (for *global* solutions, for simplicity). The underlying object is then R ; there is a standard vector field $\frac{\partial}{\partial x}$ on R , which in our context may be seen as the exponential adjoint $\hat{+}$ of the addition map $+$: $R \times D \rightarrow R$. Also, there is a standard 2ODE z (actually a spray). It is the "zero spray" z on R , which we now describe in synthetic terms. Given $t \in R^D$. Then t is of form $d \mapsto a + bd$ for unique a, b . We define $z(t)$ to be the map $D_2 \rightarrow R$ given by the "same formula", i.e.

$$\delta \mapsto a + b\delta + 0\delta^2 \text{ for } \delta \in D_2.$$

The notion of (global) solution of an ODE is of course not anything that can be negotiated; for a vector field (1ODE) $X : M \rightarrow M^D$, a solution of it is a map $y : R \rightarrow M$ such that for any $x \in R$

$$\dot{y}(x) = X(y(x)),$$

where $\dot{y}(x)$ denotes the tangent vector $D \rightarrow M$ given by $d \mapsto y(x + d)$. But the map $R \rightarrow M^D$ which to $x \in R$ associates $\dot{y}(x)$ is of course nothing but the composite

$$R \xrightarrow{\hat{+}} R^D \xrightarrow{y^D} M^D,$$

and therefore one immediately sees that a global solution of the 1ODE X is the same thing as a homomorphism from $(R, \frac{\partial}{\partial x}) = (R, \hat{+})$ to (M, X) .

A (global) solution of a 2ODE $\xi : M^D \rightarrow M^{D^2}$ is similarly a map $y : R \rightarrow M$ such that for any $x \in R$,

$$\ddot{y}(x) = \xi(\dot{y}(x)),$$

where now $\dot{y}(x)$ denotes the map $D_2 \rightarrow M$ given by $\delta \mapsto y(x + \delta)$. It is *not* the case that solutions in this sense are homomorphisms from (R, z) , in general; this is, as we shall see, rather a characteristic property of sprays, see Propositions 9 and 10 below. Consider the diagram

$$\begin{array}{ccccc}
 & & R^{D_2} & \xrightarrow{y^{D_2}} & M^{D_2} \\
 & \nearrow \hat{\dagger} & \uparrow z & & \uparrow \xi \\
 R & \xrightarrow{\hat{\dagger}} & R^D & \xrightarrow{y^D} & M^D.
 \end{array}$$

Here the lower composite $R \rightarrow M^D$ is just \dot{y} , whereas the upper composite is \ddot{y} . The left hand triangle commutes, by construction of z . To say that the total diagram commutes is to say that y is a solution of the 2ODE ξ , whereas to say that the square commutes is to say that y is a homomorphism of 2ODEs. Hence, obviously, homomorphisms are always solutions. For sprays, we have conversely:

Proposition 9 *Let ξ be a spray on M . Then a map $y : R \rightarrow M$ is a solution iff it is a homomorphism $(R, z) \rightarrow (M, \xi)$.*

Proof. Assume y is a solution. So the outer diagram commutes, equivalently, the square commutes for tangents $\in R^D$ of the form $\hat{\dagger}(a)$, i.e. for tangents of form $[d \mapsto a + d]$.

Any tangent $t \in R^D$ is of form $d \mapsto a + bd$, and such may be seen as $b \cdot [d \mapsto a + d]$. Since all maps in the square are equivariant with respect to the action of R, \cdot (for ξ , this is the spray assumption), it therefore follows that the square commutes for any $t \in R^D$.

To get a converse result, we need to assume about the 2ODE ξ on M that every $t \in M^D$ is of form $\dot{y}(0)$ for some solution. (Except for the fact that we are talking about *global* solutions, which is really just for simplicity of formulation, this is not a strong assumption, or rather, is a version of the existence theorem for solutions of differential equations.)

Proposition 10 *Let ξ be a 2ODE on M , satisfying the assumption just mentioned. If all solutions of ξ are homomorphisms $(R, z) \rightarrow (M, \xi)$, then ξ is a spray.*

Proof. Given $t \in M^D$, we represent it in the form \dot{y} for some solution. By assumption, the square in the above diagram commutes, and t is in the image of the map y^D . Now use that the maps (except possibly ξ) in the square are (R, \cdot) -equivariant, to conclude that $\xi(b \cdot t) = b \cdot \xi(t)$.

We shall finally present a result, which is of course nothing but a diagrammatic rendering of the classical method of solving 2ODEs on a manifold M , namely to solve the corresponding 1ODEs on the Tangent bundle $TM = M^D$. We describe the process that transforms a 2ODE ξ on M to a 1ODE $\tilde{\xi}$ on M^D (this is really the same as the functor q^* considered earlier).

In fact, for a given 2ODE $\xi : M^D \rightarrow M^{D_2}$, the 1ODE $\tilde{\xi}$ on M^D is the top line in the diagram

$$\begin{array}{ccccc}
 M^D & \xrightarrow{\xi} & M^{D_2} & \xrightarrow{M^+} & M^{D \times D} \cong (M^D)^D \\
 \uparrow \dot{y} & & \nearrow \ddot{y} & & \uparrow (\dot{y})^D \\
 R & \xrightarrow{\hat{+}} & & & R^D.
 \end{array}$$

Here the square (which does not involve ξ) commutes, because of associativity and commutativity of $+$, as the reader may verify by working with elements. The triangle on the left commutes iff y is a solution of the 2ODE ξ ; and the total diagram commutes iff \dot{y} is a homomorphism of vector fields $(R, \hat{+}) \rightarrow (M^D, \tilde{\xi})$. If the triangle commutes, then so does the total diagram, and the converse conclusion holds if the map M^+ is monic (which is the case if M is infinitesimally linear). We summarize:

Proposition 11 *Let $\xi : M^D \rightarrow M^{D_2}$ be a 2ODE on M , and $\tilde{\xi} : M^D \rightarrow M^{D \times D}$ the corresponding 1ODE on M^D . Given a map $y : R \rightarrow M$. If y is a solution of the 2ODE ξ , $\dot{y} : R \rightarrow M^D$ is a solution of the 1ODE $\tilde{\xi}$. The converse holds, if M^+ is monic.*

(Of course, not every solution of $\tilde{\xi}$ comes from a solution of ξ , since not every $R \rightarrow M^D$ is of form \dot{y} .)

9 Forces

Given two 2ODEs on an object M ,

$$\xi_1, \xi_2 : M^D \rightarrow M^{D_2},$$

we get for any $t \in M^D$ (viewed as a map $t : D \rightarrow M$), two maps $D_2 \rightarrow M$, namely $\xi_1(t)$ and $\xi_2(t)$, and these two maps have same restriction to $D \subseteq D_2$ (namely t). This is the condition when their *strong difference* $\xi_1(t) \dot{-} \xi_2(t)$ may be formed; we remind the reader of the notion of strong difference, which exist in many guises. Classically, it was introduced by Kolar [21], in the synthetic context by [17], and the specific version needed here may be found in [29]. The generality is the following. Given two 2-jets τ_1 and $\tau_2 : D_2 \rightarrow M$, with same restriction, say s to $D \subseteq D_2$, one may provided M is infinitesimally linear) construct a 1-jet (tangent vector) $\tau_1 \dot{-} \tau_2 : D \rightarrow M$, and conversely, given a 2-jet $\tau : D_2 \rightarrow M$ and a 1-jet $t : D \rightarrow M$, one may form a new 2-jet $\tau \dot{+} t : D_2 \rightarrow M$, and these data make the set of 2-jets restricting to a given $s : D \rightarrow M$ into a translation space (affine space) over the tangent vectorspace $T_x(M)$ (where x is the base point $s(0)$).

Applying this construction to $\xi_1(t)$ and $\xi_2(t)$ gives a tangent vector $\xi_1(t) \dot{-} \xi_2(t)$ with same base point x as t , and the law

$$t \mapsto \xi_1(t) \dot{-} \xi_2(t)$$

defines an endomap $\phi(\xi)_x$ of the tangent space $T_x M$ (not necessarily linear). Letting t range over all of $TM = M^D$, we thus get a map $\xi_1 \dot{-} \xi_2 : M^D \rightarrow M^D$ over M , i.e. a force system, in the sense of Theorem 5. Similarly, a 2ODE $\xi : M^D \rightarrow M^{D_2}$ and a force system $\phi : M^D \rightarrow M^D$ may be added to give a new 2ODE $\xi \dot{+} \phi$.

If ξ_2 is a spray (inertia), $\xi_1 \dot{-} \xi_2$ is, according to Lawvere [25] the *force* of ξ_1 with respect to ξ_2 .

10 Calculus Aspects

We intend here to make translations of the differential equations considered here, and those of the Calculus Courses we teach; this translation is appropriate for the case where the underlying manifold M is the line R . Then the

bundles R^D and R^{D^2} are identified with $R \times R$ and $R \times R \times R$, respectively, via the isomorphisms

$$R \times R \rightarrow R^D \text{ given by } (a, b) \mapsto [d \mapsto a + bd],$$

respectively

$$R \times R \times R \rightarrow R^{D^2} \text{ given by } (a, b, c) \mapsto [\delta \mapsto a + b\delta + c\delta^2].$$

(For the latter, another natural choice would be to let the last term be rather $c/2 \cdot \delta^2$ - that would save some factors 2 later on.)

A map $\xi : R^D \rightarrow R^{D^2}$ is, under the indicated identifications, a map $R^2 \rightarrow R^3$; to say that it is a 2ODE, i.e. that it splits the restriction map $R^{D^2} \rightarrow R^D$ (which under the identifications is just $(a, b, c) \mapsto (a, b)$) is to say that it is of the form $(a, b) \mapsto (a, b, g(a, b))$, for some function $g : R^2 \rightarrow R$, (on which there are no further restrictions). The zero spray on R considered earlier is the case where $g(a, b)$ is constant zero.

Also, a "general force" $\phi : R^D \rightarrow R^D$ is in a similar way given in the form

$$(a, b) \mapsto (a, f(a, b))$$

for a unique map $f : R^2 \rightarrow R$, on which there are no further restrictions.

Proposition 12 *A 2ODE ξ on R , given, as above, by a function $g : R^2 \rightarrow R$ is a spray if and only if the function g is of form $g(x, y) = h(x)y^2$ for some function $h : R \rightarrow R$.*

Proof. We first have to identify the multiplicative action of R on the bundles R^D and R^{D^2} , under the identifications. For $\tau \in R^{D^2}$, and $\lambda \in R$, $(\lambda \cdot \tau)(\delta) = \tau(\lambda\delta)$, so if $\tau(\delta) = a + b\delta + c\delta^2$, we have $(\lambda \cdot \tau)(\delta) = a + \lambda b\delta + \lambda^2 c\delta^2$, whence (under the identifications)

$$\lambda \cdot (a, b, c) = (a, \lambda b, \lambda^2 c);$$

and similarly $\lambda \cdot (a, b) = (a, \lambda b)$. The homogeneity condition for a spray ξ , for $t \in R^D$ given by (a, b) , say,

$$\xi(\lambda \cdot t) = \lambda \cdot \xi(t)$$

then translates into

$$(a, \lambda b, g(a, \lambda b)) = (a, \lambda b, \lambda^2 g(a, b));$$

if this is to hold for all a and b , g must be of the form claimed, as is seen by putting $b = 1$.

Consider two 2-jets $\tau_i : D_2 \rightarrow R$ ($i = 1, 2$) with same restriction to $D \subseteq D_2$. Under the identifications, they are given by (a, b, c_1) and (a, b, c_2) . Their strong difference $\tau_1 - \tau_2$ is then the tangent $D \rightarrow R$ given by the $(a, c_1 - c_2)$, i.e. $[d \mapsto a + (c_1 - c_2)d]$. It follows that if we have two ODEs ξ_1 and $\xi_2 : R^D \rightarrow R^{D^2}$, then their force $\phi(\xi_1, \xi_2)$, applied to a given tangent vector $t = (a, b) \in R^D$, gives the tangent vector $(a, g_1(a, b) - g_2(a, b))$, where $g_i : R^2 \rightarrow R$ corresponds to ξ_i .

It follows that the force $\phi(\xi, z)$, where z is the zero spray, is given by the same function $g : R^2 \rightarrow R$ as ξ itself.

For an arbitrary M , we can talk about “square-homogeneous” forces, i.e. maps $\phi : M^D \rightarrow M^D$ over M which satisfy

$$\phi(\lambda \cdot t) = \lambda^2 \cdot \phi(t)$$

for arbitrary $t \in M^D$ and $\lambda \in R$. The following result is also easy to prove for an arbitrary infinitesimally linear M instead of R .

Proposition 13 *Let ξ and ζ be 2ODEs on R , and assume ζ is a spray. Then ξ is a spray if and only if the force $\phi(\xi, \zeta)$ is square-homogeneous.*

Proof. Let ξ be given by $g : R^2 \rightarrow R$; ζ is given by a function of form $(x, y) \mapsto h(x)y^2$, by the Proposition 12 above, which clearly is square-homogeneous (and any function $R^2 \rightarrow R^2$ square-homogeneous in the second variable is of this form). The assertion then is that $g(x, y)$ is square homogeneous in y if and only if $g(x, y) - h(x)y^2$ is, which is evident.

We next investigate the correspondence between 2ODEs, (version 1, as considered above, and the 2ODEs, version 2, i.e. vector fields $\tilde{\xi} : M^D \rightarrow (M^D)^D \cong M^{D \times D}$, for the case $M = R$. The codomain $R^{D \times D}$ is in this case identified with R^4 , via

$$(a, b_1, b_2, e) \mapsto [(d_1, d_2) \mapsto a + b_1 d_1 + b_2 d_2 + e d_1 d_2].$$

Then a 2-jet $D_2 \rightarrow R$ given by (a, b, c) goes by composition with the addition map $D \times D \rightarrow D_2$ to the map

$$(d_1, d_2) \mapsto a + b d_1 + b d_2 + c(d_1 + d_2)^2 = a + b d_1 + b d_2 + 2c d_1 d_2,$$

which under the identifications is given by the 4-tuple $(a, b, b, 2c)$.

Finally, we shall look at the 2ODEs as given in the Calculus textbooks; here, the generality is that one is given an equation of form

$$y''(x) = k(x, y', y'').$$

In this form, they only fall under the viewpoint here provided that k does not depend explicitly on x . (If x is thought of as time, this condition is called: the equation is *autonomous*; to deal with the non-autonomous case in our context would, as usual, involve expanding the base space by one dimension, “making time spatial”.)

So let us consider the autonomous case $y''(x) = k(y, y')$, where k is an arbitrary function $R^2 \rightarrow R$. We claim that this equation corresponds to a 2ODE $\xi : R^D \rightarrow R^{D^2}$ in our sense, identified, as above, with a function $g : R^2 \rightarrow R$, provided that $k = 2g$. The term ‘corresponds’ is meant in the sense that the notion of solution is the same. To see this, we note that $y''(x)$ for an arbitrary function $y(x)$ is determined by validity of the equation (Taylor expansion)

$$y(x + \delta) = y(x) + y'(x)\delta + \frac{y''(x)}{2}\delta^2$$

for all $\delta \in D_2$. Since $\ddot{y}(x) \in R^{D^2}$ is exactly this expression considered as a function of δ , it follows that the ‘coordinates’ $\in R^3$ of $\ddot{y}(x)$ is the triple $(y(x), y'(x), 1/2y''(x))$.

So if ξ is given by a function g in two variables, as above, the equation $\ddot{y}(x) = \xi(y(x), \dot{y}(x))$ is tantamount to

$$(y(x), y'(x), 1/2y''(x)) = (y(x), y'(x), g(y(x), y'(x))),$$

or, equivalently, that

$$y''(x) = 2g(y(x), y'(x)),$$

that is, is a solution of the Calculus type equation, with $k = 2g$, as claimed.

So $y(x)$ is a solution in the Calculus sense iff

$$y(x + \delta) = y(x) + y'(x)\delta + 1/2k(y(x), y'(x))\delta^2.$$

This is to be compared with the condition we considered earlier, $\ddot{y}(x) = \xi(\dot{y}(x))$.

11 Further Topos Theoretic Aspects

Proposition 14 *Let \mathbf{E} be a locally connected (resp. locally connected) Grothendieck topos, and let $M = (T \dashv G)$ be an adjoint monad on it. Then the category of T -algebras ($\cong G$ -coalgebras) is also a locally connected (resp. locally connected) Grothendieck topos.*

Proof. We know that the category of T -algebras on \mathbf{E} is a Grothendieck topos, by Proposition 6; the notation for the structural functors to *Sets* appear in the following diagram:

$$\begin{array}{ccc}
 T - \text{Alg}(\mathbf{E}) & & \mathbf{E} \\
 \uparrow \Delta & & \downarrow \pi_0 \\
 \text{Sets} & & \text{Sets} \\
 \downarrow \Gamma & & \uparrow \Delta_0 \\
 \text{Sets} & & \text{Sets} \\
 & & \downarrow \Gamma_0
 \end{array}$$

We have to show that Δ has a left adjoint Π . The proof proceeds by giving an explicit description of Δ . Indeed,

Claim 1.

$$T(\Delta_0 S) = T(1) \times \Delta_0 S \text{ and } \Delta S = [T(\Delta_0 S) \xrightarrow{\pi_2} \Delta_0 S].$$

The first follows from the fact that T has a right adjoint:

$$T(\Delta_0 S) = T(\coprod_S 1) = \coprod_S T(1) = T(1) \times \Delta_0 S$$

where $\Pi_0 \dashv \Delta_0 \dashv \Gamma_0$ is the essential geometric morphism of \mathbf{E} .

The second follows from the first, noticing that $\Delta(1) = \mathbf{1} = [T(1) \rightarrow 1]$, since Δ is lex. Therefore

$$\begin{aligned}
 \Delta S &= \Delta \coprod_S 1 = \coprod_S \Delta 1 = \coprod_S [T(1) \rightarrow 1] = [\coprod_S T(1) \rightarrow \coprod_S 1] = \\
 &= [T(1) \times \Delta_0 S \rightarrow \Delta_0 S].
 \end{aligned}$$

Claim 2. Δ preserves inverse limits.

Since Δ is lex, it is enough to show that Δ preserves arbitrary products.

But

$$\Delta \prod_{i \in I} S_i = T(1) \times \Delta_0 \prod_{i \in I} S_i \xrightarrow{\pi_2} \Delta_0 \prod_{i \in I} S_i$$

$$= T(1) \times \prod_{i \in I} \Delta_0 S_i \xrightarrow{\pi_2} \prod_{i \in I} \Delta_0 S_i.$$

On the other hand, the product of the algebras $(\Delta S_i)_{i \in I}$ is constructed as the composite

$$\prod_{i \in I} \Delta S_i = T(1) \times \prod_{i \in I} \Delta_0 S_i \xrightarrow{\delta \times 1} T(1) \times \prod_{i \in I} \Delta_0 S_i \xrightarrow{\pi_2} \prod_{i \in I} \Delta_0 S_i,$$

(recalling that $\xi : TX \rightarrow X) \times (\eta : TY \rightarrow Y) = (T(X \times Y) \rightarrow TX \times TY \xrightarrow{\xi \times \eta} X \times Y)$, which is just π_2 . I.e. $\Delta \prod_{i \in I} S_i = \prod_{i \in I} \Delta S_i$. By Proposition ?? , this suffices to establish the existence of $\prod \dashv \Delta$.

Assume now that \mathbf{E} is also connected. Connectedness of $T - Alg(\mathbf{E})$ is equivalent to Δ being full and faithful. But this follows from the fact that Δ_0 is full and faithful:

$$\Delta S \rightarrow \Delta S'$$

corresponds bijectively to

$$T(1) \times \Delta_0 S \rightarrow T(1) \times \Delta_0 S'$$

commuting with the projections and $\Delta_0 S \rightarrow \Delta_0 S'$, which in turn corresponds bijectively to maps $S \rightarrow S'$, since Δ_0 is full and faithful.

References

- [1] J. Adamek and J. Rosicky, Accessible Categories, London Math. Soc. Lecture Notes Series No. 189, Cambridge Univ. Press 1994.
- [2] M. Artin, A. Grothendieck and J.-L. Verdier, SGA4, Springer Lecture Notes in Math. vol. 269 (1972).
- [3] M. Barr and C. Wells, Toposes, Triples and Theories, Springer Grundlehren vol. 278, 1985.
- [4] F. Borceux, Handbook of Categorical Algebra, Cambridge Univ. Press 1994.
- [5] M. Bunge, Relative Functor Categories and Categories of Algebras, Journ. Alg. 11 (1969)

- [6] M. Bunge, Models of differential algebra in the context of synthetic differential geometry, *Cahiers de Top. et Geom. Diff.* 22 (1981), 31-44
- [7] M. Bunge and P. Sawyer, On connections, geodesics and sprays in synthetic differential geometry, in *Category Theoretic Methods in Geometry* (ed. A. Kock), Aarhus Univ. Various Publ. Series 35 (1983).
- [8] S. Eilenberg and M. Kelly, *Closed Categories*, in *Proc. Conf. Categorical Alg.*, LaJolla, Springer Verlag 1966
- [9] S. Eilenberg and J. Moore, Adjoint functors and triples, *Illinois Journ. Math.* 9 (1965), 381-398.
- [10] P. Johnstone, Cartesian monads on toposes, *J. Pure Appl. Alg.* 116 (1997), 199-220.
- [11] M. Kelly, Adjunction for enriched categories, in *Reports of the Midwest Category Seminar III*, Springer Lecture Notes in Math. 106 (1969).
- [12] M. Kelly, *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Notes Series No. 64, Cambridge Univ. Press 1982.
- [13] A. Kock, Monads on Symmetric Monoidal Closed Categories, *Arch. Math.* 21 (1970), 1-10.
- [14] A. Kock, Strong Functors and Monoidal Monads, *Arch. Math.* 23 (1972), 113-120.
- [15] A. Kock, Closed Categories Generated by Commutative Monads, *J. Austr. Math. Soc.* 12 (1971), 405-424.
- [16] A. Kock, *Synthetic Differential Geometry*, London Math. Soc. Lecture Notes Series No. 51, Cambridge Univ. Press 1981.
- [17] A. Kock and R. Lavendhomme, Strong infinitesimal linearity, with applications to strong difference and affine connections, *Cahiers de Top. et Geom. Diff.* 25 (1981), 311-324.
- [18] A. Kock and G.E. Reyes, Manifolds in formal differential geometry, in *Durham Proceedings 1977*, Springer Lecture Notes 753 (1979).

- [19] A. Kock and G.E. Reyes, Connections in formal differential geometry, in Topos Theoretic Methods in Geometry (ed. A. Kock), Aarhus Univ. Various Publ. Series No. 30 (1979).
- [20] A. Kock and G.C. Wraith, Elementary Toposes, Aarhus Lecture Notes Series No. 30, 1971.
- [21] I. Kolar, On the second tangent bundle and generalized Lie derivatives, Tensor N.S. 38 (1982), 98-102.
- [22] R. Lavendhomme, Basic Concepts of Synthetic Differential Geometry, Kluwer, Dordrecht 1996.
- [23] F.W. Lawvere, Categorical Dynamics, in Topos Theoretic Methods in Geometry (ed. A. Kock), Aarhus Univ. Various Publ. Series No. 30 (1979).
- [24] F.W. Lawvere, Fractional Exponents in Cartesian Closed Categories, Manuscript, Jan. 1988.
- [25] F.W. Lawvere, Outline of Synthetic Differential Geometry, Manuscript Feb. 1998 (augmented Nov. 1998). Available from <http://www.acsu.buffalo.edu/~wlawvere/downloadlist.html>
- [26] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic, Springer Universitext, 1992.
- [27] I. Moerdijk, The classifying topos of a continuous groupoid, Trans. Amer. Math. Soc. 310 (1988) , 629-668.
- [28] I. Moerdijk and G.E. Reyes, Models for Smooth Infinitesimal Analysis, Springer Verlag 1991.
- [29] H. Nishimura, Nonlinear connections in synthetic differential geometry, J. Pure Appl. Alg. 131 (1998), 49-77.
- [30] D. Yetter, On right adjoints to exponential functors, J. Pure Appl. Alg. 45 (1987), 287-304. Corrections: J. Pure Appl. Alg. 58 (1989), 103-105.