

Commutative monads, and distributions (in a broad sense)

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The word “distribution” is used for many things, including the (Schwartz-) distributions of Functional Analysis, probability distributions of random variables, and distributions of tomato on a pizza (which is a *mass* distribution). The theory to be presented here applies in particular to all of these three cases.

The research grew out of my attempt to understand some aspects of probability theory, notably probability distributions.

I realized that a probability distribution, like the other kinds of distributions mentioned, is an example of an *extensive quantity* in a sense made explicit by Lawvere: the salient feature of a given type of extensive quantity is *covariant functoriality*.

Thus, one has a category \mathcal{E} of *spaces*; it is typically cartesian closed; one has an additive category \mathcal{A} (typically, the category of real vector spaces, to be concrete); an extensive quantity (of a given type) is a functor $Q : \mathcal{E} \rightarrow \mathcal{A}$ (with certain properties which I shall not list here). The functor Q assigns to a space $X \in \mathcal{E}$ the vector space $Q(X)$ of extensive quantities (of the given type) “varying over” X .

Some types of extensive quantities have the property that the additive category \mathcal{A} is tied to \mathcal{E} in the sense that it is a category of objects in \mathcal{E} , equipped with some further *structure*, say *R-module* structure *in* \mathcal{E} for some ring object R in \mathcal{E} .

What I understand by a *distribution* is an extensive quantity of this particular kind, namely where \mathcal{A} is the category of T -algebras for a suitable monad $T = (T, \eta, \mu)$ on \mathcal{E} .

So for a “space” $X \in \mathcal{E}$, we have the space $T(X) \in \mathcal{E}$ “of distributions over X ”; we also have a map $\eta_X : X \rightarrow T(X)$, which you should think of as associating to $x \in X$ the *Dirac distribution* δ_x at x . (For mass distributions, one must *choose* a unit of mass first; this is where a torsor aspect comes in; see [10].)

Example. Let \mathcal{E} be the category of sets. For $X \in \mathcal{E}$, let $T(X)$ be the free real vector space on X , so $T(X)$ is a vector space with X as a *basis*. (Note that $T(\mathbf{1}) = \mathbb{R}$ here.) Equivalently, $T(X)$ is the set of formal linear combinations (with real coefficients) of the elements from X . So $P \in T(X)$ may be presented as a dispersion of real numbers (“coefficients”) across X ; the coefficient which x receives is 0 for all but a finite number of x s. If all coefficients are

non-negative, and if they add up to 1, then P may be called a *probability* distribution on X ; for $P = \eta_X(x)$, x occurs with probability 1, i.e. x occurs *with certainty*.

Now this example is in one sense misleading: $T(X)$ is here viewed as an *abstract* real vector space; this in particular applies to $T(\mathbf{1}) = \mathbb{R}$, so all *cohesion* (e.g. in terms of topology) is lost. We would like to have a theory of monads T on a (cartesian closed) category \mathcal{E} of not-nec.-discrete spaces. There exist such, \mathcal{E}, T , cf. [3] Theorem 5.1.1 and [1], and in fact with $T(X)$ being the space of (Schwartz-) distributions (of compact support) on the space X , but everything with due regard to the *cohesion*. In any case, for some of the applications of the present theory, it is necessary that we consider not just the category \mathcal{E} of abstract sets, but a more general cartesian closed \mathcal{E} . This is what I shall do from now on, working axiomatically with a monad T on \mathcal{E} – but sometimes specializing to $\mathcal{E} = \mathbf{Sets}$, for illustration.

In a cartesian closed category, I write $X \pitchfork Y$ for Y^X .

1 Monads on Sets, and algebraic theories

Essentially, a monad $T = (T, \eta, \mu)$ on $\mathcal{E} = \mathbf{Sets}$ is the same as an (infinitary) algebraic theory \mathbb{T} . This is old wisdom from the 1960s, which I shall remind you of.

First, let $B = (B, \beta)$ be an algebra for T , so $\beta : T(B) \rightarrow B$ (satisfying a unit- and associative law). Then for any $X \in \mathcal{E}$, $X \pitchfork B$ inherits an algebra structure (the “coordinatewise” structure), namely

$$T(X \pitchfork B) \xrightarrow{\lambda} X \pitchfork T(B) \xrightarrow{X \pitchfork \beta} X \pitchfork B.$$

Here, λ is the (cotensorial) *strength* ([6]¹) of the endofunctor $T : \mathcal{E} \rightarrow \mathcal{E}$; for \mathcal{E} the category of sets, the “combinator” λ can be described by reinterpreting $X \pitchfork Y$ as the product of X copies of Y . In particular, we have commutative triangle

$$\begin{array}{ccc} T\left(\prod_{x \in X} Y\right) & \xrightarrow{\lambda_{X,Y}} & \prod_{x \in X} T(Y) \\ & \searrow T(pr_x) & \downarrow pr_x \\ & & T(Y) \end{array}$$

where pr_x is the projection to the x th factor.

In particular, with $Y = B$, the set $(X \pitchfork B) \pitchfork B$ inherits a T -algebra structure from (the second occurrence of) B .

Now it is a basic fact of monad theory that $T(X)$ (with structure map $\mu_X : T^2(X) \rightarrow T(X)$) is a *free* T -algebra on X , more precisely, $\eta_X : X \rightarrow T(X)$ has the well known universal property of being *initial* among maps from X into T -algebras.

We have, by “pure lambda calculus” a map $\delta : X \rightarrow (X \pitchfork B) \pitchfork B$. It associates to $x \in X$ the map $(X \pitchfork B) \rightarrow B$ given by $f \mapsto f(x)$ for $f \in (X \pitchfork B)$. Alternatively, δ is the exponential

¹The strength can also be encoded as *tensorial* strength, $T(X) \times Y \rightarrow T(X \times Y)$; see [4].

transpose of the evaluation map $X \times (X \multimap B) \rightarrow B$. (This latter description works in any CCC.) The letter “ δ ” stands for “Dirac”.

But since $(X \multimap B) \multimap B$ has a T -algebra structure inherited from that of B , and since $\eta_X : X \rightarrow T(X)$ has the universal property described, we conclude that there exists a unique T -algebra homomorphism τ making the diagram

$$\begin{array}{ccc}
 T(X) & \xrightarrow{\tau} & (X \multimap B) \multimap B \\
 \uparrow \eta_X & \nearrow \delta & \\
 X & &
 \end{array}$$

commutative.

More completely, τ should be decorated by symbols X and (B, β) . It is natural in each of them.

The set $X \multimap B = X^B$ is the set of X -tuples of elements in B . The τ is the *semantics* map: for each element $P \in T(X)$, one gets an X -ary operation on B , meaning a map $X \multimap B \rightarrow B$, i.e. an element in $(X \multimap B) \multimap B$, namely $\tau(P)$.

For instance, when T is the free-commutative-ring formation, $T(X)$ is the ring of polynomials in X variables; these are *syntactic* entities. A T -algebra B is the same as a commutative ring. If P is a polynomial in X variables, $\tau(P)$ is the polynomial *function* $B^X \rightarrow B$ to which P gives rise. This function is the *semantics* of P in B .

2 Commutative monads

If α is T -algebra structure on the set A and β is a T -algebra structure on the set B , we have a subset $A \multimap_T B \subseteq A \multimap B$ consisting of those maps $A \rightarrow B$ that happen to be T -algebra homomorphisms. (It is not in general a subalgebra.) This is an equalizer construction, and may also be performed in any cartesian closed category with equalizers.

Definition 1 *The monad T on a cartesian closed² category \mathcal{E} is commutative if for all T -algebras B and C , the subobject*

$$B \multimap_T C \subseteq B \multimap C$$

is a subalgebra.

More compact, equivalent, conditions are given in [10], Theorem 4 (which in turn summarizes my 1970-72 papers [4], ..., [8]).

Example: The monad whose algebras are abelian groups is commutative: the set of group homomorphisms between two abelian groups A and B is a group, in fact a subgroup

²most of the following works equally well in any symmetric monoidal closed category

of $A \multimap B$ (whose addition is inherited from that of B). – The monad whose algebras are groups is not commutative, for the same reason.

If B is T -algebra, we have the subset $(X \multimap B) \multimap_T B \subseteq (X \multimap B) \multimap B$. Since the T -algebra structure on $(X \multimap B)$ is coordinatewise, it is clear that $\delta : X \rightarrow (X \multimap B) \multimap B$ factors through this subset. So if this subset is a subalgebra, the universal property of $\eta_X : X \rightarrow T(X)$ gives that $\tau : T(X) \rightarrow (X \multimap B) \multimap_T B \subseteq (X \multimap B) \multimap B$.

So if the monad is commutative, we have (for any object $X \in \mathcal{E}$ and any T -algebra B) that the semantics map τ corestricts to a map, likewise denoted τ :

$$T(X) \xrightarrow{\tau} (X \multimap B) \multimap_T B.$$

and it is a T -homomorphism, since the original τ was so, by construction. (One may see this τ as a canonical comparison between an abstract distribution notion, as encoded by T , and the distribution notion in the double-dualization/Schwartz paradigm.) We also consider the exponential transpose of this map

$$T(X) \times (X \multimap B) \xrightarrow{\hat{\tau}} B.$$

As a function in two variables, $\hat{\tau}$ has good properties; it is

- a T -homomorphism in the first variable, since τ is a T -homomorphism; and
- a T -homomorphism in the second variable, since τ factors through the subset $(X \multimap B) \multimap_T B$ consisting of T -homomorphism.

The terminology becomes a little heavy now! We would like to just say: $\hat{\tau}$ is T -bilinear, and similarly, use “ T -linear” for “ T -homomorphism”. Since $\hat{\tau}$ is T -bilinear in this sense, we shall use a bilinear-looking notation for it, thus we write

$$\langle P, \phi \rangle \text{ for } \hat{\tau}(P, \phi),$$

for $P \in T(X)$ and $\phi \in X \multimap B$ (so $\phi : X \rightarrow B$). (In distribution theory in the sense of functional analysis, the notation $\int_X \phi \, dP$, or even $\int_X \phi(x) \, dP(x)$, is also used for $\langle P, \phi \rangle$.) We call this the *canonical pairing* or the *integration pairing*. In this Schwartz paradigm, ϕ would be called a *test function*.

Lemma 1 *If $\phi \in X \multimap B$, then $\phi : X \rightarrow B$ is a map, and $\langle P, \phi \rangle \in B$ may alternatively be described as the value on P of the map*

$$T(X) \xrightarrow{T(\phi)} T(B) \xrightarrow{\beta} B.$$

Proof. As functions of $P \in T(X)$, both expressions are T -linear in P ; hence by the universal property of η_X , it suffices to see that we get equality for $P = \eta_x, x \in X$. On $\eta_X(x)$, $\beta \circ T(\phi)$ yields, by naturality of η just $\beta(\eta_B(\phi(x)))$ which is $\beta(x)$, since $\beta \circ \eta_B = id_B$. On the other hand $\langle \eta_X(x), \phi \rangle = \delta(x)(\phi) = \phi(x)$, by definition of τ , and hence the pairing, in terms of $\delta : X \rightarrow (X \multimap B) \multimap B$.

3 Monoidal structure

For a commutative monad T on \mathcal{E} , the notion of T -bilinear map $A \times B \rightarrow C$ makes sense and has good properties, cf. [7] (recall linear := T -algebra homomorphism) whenever A, B , and C are equipped with T -algebra structure. (It is most economically expressed in terms of tensorial strength, $T(X) \times Y \rightarrow T(X \times Y)$.)

We have

Proposition 1 *Let $B = (B, \beta)$ be a T -algebra. Then any map $f : X \times Y \rightarrow B$ extends uniquely over $\eta_X \times \eta_Y : X \times Y \rightarrow T(X) \times T(Y)$ to a T -bilinear $\bar{f} : T(X) \times T(Y) \rightarrow B$.*

Proof. See [10].

In particular, apply this to the case where B is $T(X \times Y)$ and f is $\eta_{X \times Y}$. Then we get a T -bilinear map $T(X) \times T(Y) \rightarrow T(X \times Y)$, which we denote $\otimes_{X,Y}$:

$$T(X) \times T(Y) \xrightarrow{\otimes_{X,Y}} T(X \times Y).$$

In fact, in [4] it is proved that this \otimes , together with $\eta_{\mathbf{1}} : \mathbf{1} \rightarrow T(\mathbf{1})$, makes T into a monoidal functor. (Here $\mathbf{1}$ denotes the terminal object.) This implies e.g. that if M is a monoid, then $T(M)$ acquires a monoid structure as well, with multiplication

$$T(M) \times T(M) \xrightarrow{\otimes_{M,M}} T(M \times M) \xrightarrow{T(\bullet)} T(M),$$

where \bullet is the multiplication of M ; and similarly, if the monoid M acts on V , then $T(M)$ acts on $T(V)$. Since \otimes is T -bilinear, we conclude that the multiplication on $T(M)$, as well as its action on $T(V)$, are T -bilinear.

Actually, the monoidal structure \otimes on the functor T makes (T, η, μ) into a symmetric monoidal monad (see [8]), and this implies that if the monoid M is a commutative monoid, then so is $T(M)$.

Now $\mathbf{1}$ carries a (unique, and trivial) monoid structure. It is commutative; and this trivial monoid acts, trivially, on any object V . Therefore, $T(\mathbf{1})$ carries a canonical (but not trivial, in general) commutative monoid structure, and this monoid acts on any object of the form $T(V)$. For naturality reasons, for any map $f : V \rightarrow V'$, the map $T(f) : T(V) \rightarrow T(V')$ is equivariant with respect to the action by $T(\mathbf{1})$. We have, however, a better result:

Proposition 2 *Any T -algebra B carries a canonical action by $T(\mathbf{1})$, and any T -linear map is equivariant with respect to this action.*

Proof sketch. To construct the action, the main point is to prove that the map $\mu_X : T^2(X) \rightarrow T(X)$ is equivariant, for any X . Then the parallel pair in the coequalizer diagram

$$T^2(B) \begin{array}{c} \xrightarrow{\mu_B} \\ \xrightarrow{T(\beta)} \end{array} T(B) \xrightarrow{\beta} B$$

are equivariant, and therefore, the action descends along β (= the structure map of the T -algebra B). The proof of equivariance of T -linear maps uses the same kind of coequalizers.

To prove equivariance of μ_X means to prove that equality of two maps $T^2(X) \times T(\mathbf{1}) \rightarrow T(X)$; both the maps to be compared are T -bilinear, so by Proposition 1, it suffices to see that they agree when precomposed by $\eta_{TX} \times \eta_{\mathbf{1}}$; this is moderately easy, see Proposition 11 in [9].

4 The scalars

We summarize: $T(\mathbf{1})$ is a commutative monoid. It acts canonically on any T -algebra, and T -linear maps between such are equivariant for the action. Therefore, we may think of the monoid $T(\mathbf{1})$ as the (multiplicative) monoid of scalars, for “ T -linear spaces”, meaning T -algebras. Therefore, we like to denote it R , or $(R, \cdot, 1)$, where \cdot is the multiplication, and 1 is the element picked out by $\eta_{\mathbf{1}} : \mathbf{1} \rightarrow T(\mathbf{1})$. Thus we have a faithful functor from the category \mathcal{E}^T of T -linear spaces to the category of spaces equipped with an action by R .

We would like also an *addition* on R , so as to make it into a ring, or at least, a rig. This depends on certain *properties* of the category \mathcal{E} and of the monad T ; (not an added *structure* on \mathcal{E}, T); namely: \mathcal{E} should have finite coproducts; and T should take finite coproducts into products, in a sense which can be made precise, see [2] and [9]. In particular, $T(\emptyset) = \mathbf{1}$, in which case this T -linear space is easily seen to be a zero object in the category \mathcal{E}^T of T -linear spaces (= T -algebras).

Under these assumptions, every T -linear space B acquires a T -linear addition map $+ : B \times B \rightarrow B$; it makes B into an abelian monoid, and any T -linear map preserves addition. Also, T -bilinear maps are bi-additive, in an evident sense. This in particular applies to the multiplication $R \times R \rightarrow R$, and to the multiplicative action of R on T -linear spaces.

Thus, $(R, \cdot, 1, +, 0)$ is canonically a rig (= commutative semiring with unit), and any T -linear space is canonically a module over this rig. Also, T -linear maps are equivariant and preserve addition, so that there is a faithful forgetful functor

$$\mathcal{E}^T \rightarrow R\text{-Mod.}$$

This justifies the “linear” terminology. Note that we are not asserting that the functor is full, in other words, T -linear is stronger than R -linear. In the applications, one rather has that “ T -linear” means “ R -linear *and* continuous”.

5 Totals and expectations

We contend that a commutative monad T encodes a notion of *distribution* in a broad sense; thus, we use the terminology that an element $P \in T(X)$ is a *distribution over the space X* .

A distribution of tomato over a pizza has a *total*: the total amount of tomato utilized. This is encoded in the covariant functorality: if $P \in T(X)$ is a distribution of something over

the space X , its *total* $\text{tot}(P)$ is $T(!_X)(P)$, where $!_X$ denotes the unique map from X to $\mathbf{1}$; thus $\text{tot}(P)$ is a scalar³.

It is clear that if $f : X \rightarrow Y$ is any map, and $P \in T(X)$, then

$$\text{tot}(T(f)(P)) = \text{tot}(P).$$

For, $!_Y \circ f = !_X$.

For $x \in X$, naturality of η w.r.to $!_X$ gives

$$\text{tot}(\eta(x)) = 1 \tag{1}$$

where η denotes $\eta_X : X \rightarrow T(X)$. (Recall that 1 is $\eta_{\mathbf{1}} : \mathbf{1} \rightarrow T(\mathbf{1}) = R$.)

It is an easy consequence of the naturality of the \otimes maps that if P_1 and P_2 are distributions over the spaces X_1 and X_2 , respectively, (so $P_i \in T(X_i)$), then

$$\text{tot}(P_1 \otimes P_2) = \text{tot}(P_1) \cdot \text{tot}(P_2);$$

after all, the multiplication \cdot on R may be identified with $\otimes_{\mathbf{1}, \mathbf{1}}$, modulo the identification of $\mathbf{1} \times \mathbf{1}$ with $\mathbf{1}$.

Another aspect of covariant functoriality is the notion of *marginal distributions* of a distribution on a product space: if $P \in T(X_1 \times X_2)$, we get two marginal distributions $P_i := T(\text{proj}_i)(P) \in T(X_i)$, $i = 1, 2$. Each of them has the same total as P .

We may ask: if $P \in T(X_1 \times X_2)$ has marginal distributions P_1 and P_2 , do we then have $P = P_1 \otimes P_2$? A necessary condition for this to happen is, by the above, that the scalar $\text{tot}(P)$ is multiplicatively idempotent, so, for the purposes of interest here, $\text{tot}(P) = 0$ or $\text{tot}(P) = 1$.

Similarly, we may ask, given $P_i \in T(X_i)$ ($i = 1, 2$), are the P_1, P_2 the marginal distributions of $P_1 \otimes P_2$? A sufficient condition for this to be so is that $\text{tot}(P_1) = \text{tot}(P_2) = 1$. This follows from the observation that there is an ‘‘affine’’ submonad T_0 of T , to which Theorem 2.1 in [7] applies, see [9], Section 10. For $P \in T(X)$, $P \in T_0(X)$ iff $\text{tot}(P) = 1$.

Such P s, we may call *normalized*; any probability distributions on a space X is normalized.

For any $X \in \mathcal{L}$, we have the map

$$X \xrightarrow{!_X} \mathbf{1} \xrightarrow{\eta_{\mathbf{1}}} T(\mathbf{1}) = R;$$

the map is $\eta_{\mathbf{1}}$, picks out the multiplicative unit of the rig R . In more elementary terms, this is the function on X ‘‘with constant value $1 \in R$ ’’, or $1_X(x) = 1$ for all $x \in X$.

Recall the ‘‘integration’’ pairing $T(X) \times (X \pitchfork B) \rightarrow B$, which we have for any T -linear space B . In particular, for $B = R$, we have

Proposition 3 *Let $P \in T(X)$. Then*

$$\text{tot}(P) = \langle P, 1_X \rangle.$$

³or becomes so, after a choice of unit mass, for the case of mass distributions, say; cf. the section on *torsors* over T in [10].

Thus, in the “integral” notation for the pairing,

$$\text{tot}(P) = \int_X 1_X dP = \int_X 1 dP(x).$$

Proof. Use the alternative formulation of the pairing (last three lines of Section 2). Then the result follows from the fact that $T(\eta_1)$ followed by the structure map μ_1 of $R = T(\mathbf{1})$ is the identity map.

Distributions on the line R

We shall now in particular consider distributions on the space (line) R of scalars (more generally, on any T -algebra $B = (B, \beta)$). Probability distributions of random variables on an outcome space Ω are examples. Such a distribution has an *expectation*. We have in fact a notion of expectation $E(P) \in B$ for any $P \in T(B)$; it is defined in terms of the integration pairing, as

$$E(P) = \langle P, id_B \rangle = \int_B id_B dP = \int_B x dP(x).$$

Thus, we have that E is a map $T(B) \rightarrow B$; this is a map $T(B) \rightarrow B$.

Proposition 4 *The map $E : T(B) \rightarrow B$ equals β .*

The proof is similar to the proof of the previous Proposition; just note that $T(id_B)$ is itself the identity map of $T(B)$.

From the Proposition and a standard equation relating μ and η , we therefore get

$$E(\eta(x)) = x. \tag{2}$$

In the following, we consider $(B, \beta) = (R, \mu_1)$, and η denotes $\eta_R (= \eta_{T(\mathbf{1})})$.

Using the multiplicative monoid structure on R , we not only have the identity map x , but also the squaring map x^2 , the cubing map x^3 , \dots . The *n th moment* of a distribution $P \in T(R)$ is defined in terms of the pairing as $\langle P, x^n \rangle$.

It becomes more interesting when we bring in the addition $+$ of R . Then we have the map “convolution along $+$ ”,

$$T(R) \times T(R) \xrightarrow{\otimes_{R,R}} T(R \times R) \xrightarrow{T(+)} T(R)$$

which we denote by $*$. It is T -bilinear (hence R -bilinear), and associative and commutative (by arguments similar to those of Section 4).

Note that, by the very definition of \otimes , we have that $\eta(x) * \eta(y) = \eta(x+y)$ for x and y in R .

Proposition 5 *Let P and Q be distributions on R , i.e. $P \in T(R)$ and similarly for Q . Then*

$$E(P * Q) = E(P) \cdot \text{tot}(Q) + \text{tot}(P) \cdot E(Q).$$

Proof. Since $*$ and \cdot are T -bilinear, and E and tot are T -linear, each of the two terms on the right depend in a T -bilinear way on P, Q ; since also $+$ is T -linear, the whole right hand side depends in a T -bilinear way on P, Q , and therefore, by Proposition 1, it suffices to prove the equation for P, Q of the form $\eta(x), \eta(y)$. Then the left hand side gives $E(\eta(x+y)) = x+y$, and the right hand side gives $x \cdot 1 + 1 \cdot y$ which is likewise $x+y$.

In particular, if P and Q are *normalized* distributions, meaning having total 1, then $E(P * Q) = E(P) + E(Q)$. This therefore in particular applies to probability distributions. In probability theory, P and Q are distributions of independent random variables, and the Proposition gets the verbal formulation that the expectation of the sum is the sum of the expectations. This even holds without assuming independence, and in our context, this is shown in [10].

Recall that a translation map $\alpha : R \rightarrow R$ is a map of the form $x \mapsto a+x$ for some $a \in R$. For this α , $T(\alpha)(P) = \eta(a) * P$; for, both sides depend in a T -linear way on P , so it suffices to check it for P of the form $\eta(x)$; then both sides give $\eta(a+x)$.

Proposition 6 *If P is a normalized distribution on R , then for any translation map $\alpha : R \rightarrow R$,*

$$E(T(\alpha)(P)) = \alpha(E(P)).$$

Proof. We have $T(\alpha)(P) = \eta(a) * P$, and by the previous Proposition, $E(\eta(a) * P) = E(\eta(a)) + E(P) = a + E(P)$.

This gives e.g. that the center of gravity of a mass distribution on a line R is invariant under translations, see [10]

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