THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Department of Mathematical Sciences | University of Aarhus
Ny Munkegade | Building 1530 | 8000 Aarhus | Denmark
Phone + 45 89 42 35 15 | **Fax** + 45 86 13 17 69
Mail: thiele@imf.au.dk | **Web:** www.thiele.au.dk

Surface area and rose of direction from digital images

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Project

Digital Stereology

Project members

Eva B. Vedel Jensen, Markus Kiderlen, Gunnar Hellmund, Johanna Ziegel (ETH Zürich)

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Description of problem

We want to estimate the surface area $S(X)$ of a set $X \subset \mathbb{R}^d$ from a digitization of X. One common model for digitization is the **Gauss digitization** $\tilde{X}_t$ obtained by considering all points of the scaled standard lattice $t\mathbb{Z}^d$, $t > 0$, that lie in X. Representing a set in Euclidean space by a finite one means of course a considerable loss of information, even if the resolution $1/t$ of the lattice is very large. Common algorithms yield to errors up to 40 % in 2D; see an example. The corresponding algorithm in 3D may have an error up to 73 %.

Surface area estimation from $2 \times 2 \times 2$-configurations

We developed an algorithm to estimate $S(X)$ for a three-dimensional set X with an error of at most 4 %, as $t \to 0+$. The estimation procedure is based on enumerating occurrences of $2 \times 2 \times 2$–configurations $\mathbf{C}$; defined here. Asymptotically (with increasing resolution of the digitization), appropriate normalizations of the counts $\#\mathbf{C}_t$ are known to converge to integrals involving the surface are measure $S(X, \cdot)$ of X. To be more precise, we have

$$t^{d-1} E\#\mathbf{C}_t \to \int_{S^{d-1}} h(\mathbf{C}, -u)\, dS(X, u), \qquad t \to 0+,$$

where E means the usual expectation and $h(\mathbf{C},\cdot)$ can easily be calculated from the configuration, cf. Kiderlen and Rataj (2006), also for the mild assumptions on X. This relation can be exploited to show how a weighted sum of the different counts yields an estimator of $S(X)$. In the planar case, the boundary length was estimated this way in V. Jensen and Kiderlen (2003) and Kiderlen and V. Jensen (2003). The 3D case is discussed in Ziegel (2007). It should be noted that the suggested procedure is based on local configuration counts only and does not require an explicit reconstruction of the boundary of X. Therefore, implementation is straightforward and leads to extremely efficient procedures.

Refining the above described method, one can use configuration counts to estimate the surface area measure of X. When X is a random set, its rose of normal directions can be estimated this way.

An example: Boundary length from pixel approximation

Applying the **pixel approximation** (**"Integral geometric approach"**), (cf. Michielsen et al. (2002), Mecke (1993)) to the unit square X and its rotation Y by 45^o, we obtain the approximations $\hat{S}(\tilde{X}_t) = 4.2$, $\hat{S}(\tilde{Y}_t) = 5.04$, see Figure 1, where a lattice scaling of $t = 0.21$ was used.

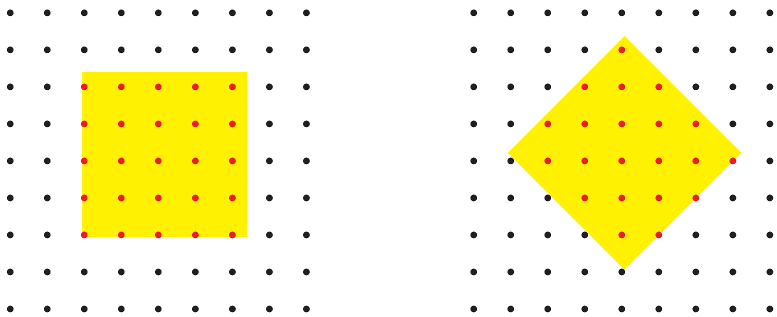


Figure 1: The unit square X (left) and its rotation Y (right).

The estimators deviate by 20 % although the true boundary length is the same for both sets. To explain this behaviour, we must understand how these estimators are obtained. As can be seen in Figure 2, the estimator is just the boundary length of a polygonal approximation P of the set. In the figures, P is indicated in red and obtained as the union of all pixels (scaled lattice squares, centered at the points of the lattice) whose midpoint is contained in the digitization. Hence, even if $t \to 0+$ the approximation $\hat{S}(\tilde{Y}_t)$ will not converge to 4, but to $4\sqrt{2}$, which yields a deviation of about 40 %. In applications, the effect is typically less expressed, but nevertheless it is far from neglectable.

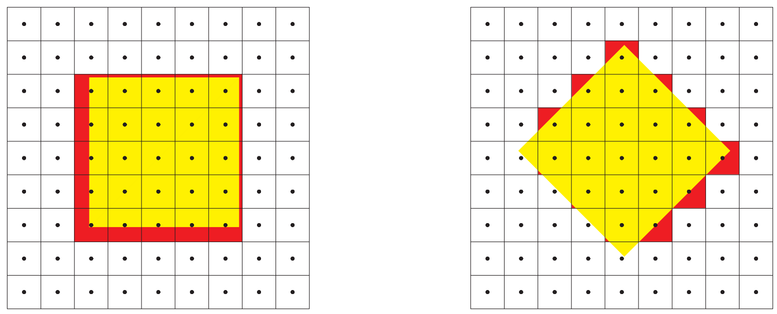


Figure 2: Pixel approximation of the unit square X (left) and its rotation Y (right).

$2 \times 2 \times 2$-configurations and the counting procedure

A $2 \times 2 \times 2$-configuration can be seen as a coloring of the vertices of the unit cube with two different colors, say red and black. Figure 3 gives an example, where red is indicating that the corresponding point is an element of $\tilde{X}_t$, and black points are in its complement.

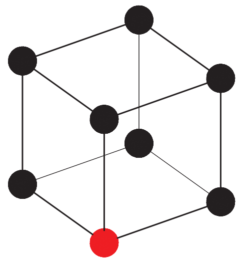


Figure 3: A $2 \times 2 \times 2$-configuration with one point in $\tilde{X}_t$.

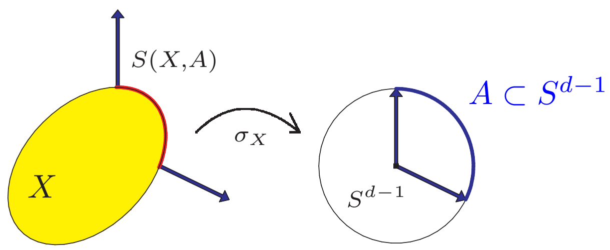
If R and B denote the sets of red and black points of $\mathbf{C}$, respectively, then

$$\mathbf{C}_t := \{x \in t\mathbb{Z}^d \mid x + tR \subset \tilde{X}_t,\ x + tB \subset \mathbb{Z}^d \setminus \tilde{X}_t\}$$

is the set of all positions of the scaled $\mathbf{C}$ in the digital image and $\#\mathbf{C}_t$ is obtained by counting all occurrences of this configuration in the image.

Definition of the surface area measure

The **surface area measure** $S(X,\cdot)$ **of** $X \subset \mathbb{R}^d$ is the image measure of the $(d-1)$-st Hausdorff measure $\mathcal{H}^{d-1}$ on ∂X under the spherical image map $\sigma_X : \partial X \to S^{d-1}$. Here we used the fact that for sufficiently regular sets X (e.g. if X is polyconvex) $\mathcal{H}^{d-1}$-almost all points $x \in \partial X$ have a unique outer unit normal $\sigma_X(x)$.



This measure is a local counterpart of $S(X)$, as its total mass is $S(X, S^{d-1}) = S(X)$.

Our publications on this topic

M. Kiderlen and J. Rataj. On infinitesimal increase of volumes of morphological transforms. *Mathematika*, 53:103–127, 2006.

M. Kiderlen and E. B. V. Jensen. Estimation of the directional measure of planar random sets by digitization. *Adv. Appl. Probab.*, 35(3):583–602, 2003.

E. B. V. Jensen and M. Kiderlen. Directional analysis of digitized planar sets by configuration counts. *J. Microsc.*, 212:158–168, 2003.

J. Ziegel. Estimation of surface area measures from digitized sets in three dimensional space. *In preparation*, 2007.

Related approaches

The above mentioned **pixel approximation** was one of the first approaches, used to estimate perimeter, surface area and also other instrinsic volumes. Mecke (1993) uses this method in the analysis of digital images from physics and Michielsen et al. (2002) discuss its connections to (continuous) integral geometry. However, as we have seen, the measurement error can be considerable if the structure under consideration is anisotropic.

Crofton's formula states that the surface area in $\mathbb{R}^d$ can be estimated by counting the number of connected components in sections of X with "randomly moved" lines. Discretizing this relation, Schladitz et al. (2006) (see also earlier work quoted there) established an estimator for the surface area measure, which is asymptotically biased. The worst case asymptotic deviation is not explicitly known.

Finally, Lindblad (2005) has established an estimator which is, like the one we introduced here, an **weighted sum of configuration counts**. His weights differ considerably from ours, and are designed for the case where X is isotropic. Under this assumption, his weights yield an asymptotically unbiased estimator with minimal mean square error. Without this assumption, the estimator is biased, but Lindblad did not consider its asymptotic behavior.

Related publications

J. Lindblad. Surface area estimation of digitized 3d objects using weighted local configurations. *Image Vis. Comp.*, 23:111–122, 2005.

K. Mecke. *Integralgeometrie in der Statistischen Physik*. Harry Deutsch, Frankfurt a. M., 1993.

K. Michielsen, H. De Raedt, and J. De Hosson. Aspects of mathematical morphology. *Adv. Imaging Electron Phys.*, 125:119–195, 2002.

K. Schladitz, J. Ohser, and W. Nagel. Measurement of intrinsic volumes of sets observed on lattices. In A. Kuba, L. G. Nyul, and K. Palagyi, editors, *13th International Conference on Discrete Geometry for Computer Imagery,*, pages 247–258. Springer, 2006.