Optimal Metrics on Compact 4-Manifolds

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Motivation

The uniformization theorem tells us that any 2-manifold admits a Riemannian metric of constant Gauss curvature.

Question

Is it possible to define a notion of "best metric" for an arbitrary n-manifold, and if so, does every manifold admit such a metric?

Discussion

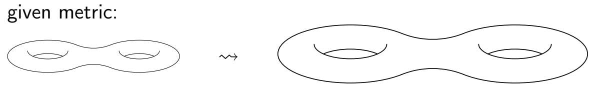
What should one require of a "best metric"?

The 2-dimensional case suggests that the definition should involve the curvature tensor. Furthermore, when a metric of constant sectional curvature exists it should certainly be called "best".

Agreeing that the flat metrics on the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ are "best", we search for metrics on other manifolds which are as "flat as possible".

Flattening a metric

On any smooth compact manifold M, there is a Riemannian metric whose curvature is uniformly close to zero. Namely, rescale any given metric:



Multiplying the metric g by a positive constant c, the length of the curvature tensor is multiplied by c^{-1}. Thus we obtain a flatter metric at the cost of a larger volume.
We need some sort of scale-invariant measure of the curvature.

L^p-norms

> *Integrating a smaller function over a larger volume yields just about the same as integrating a larger function over a smaller volume.*

This is of course not a very precise statement, but nevertheless it motivates the use of L^p-norms of the curvature tensor as a measure of the curvature.
For a smooth compact manifold M of dimension n, let $\mathcal{S}_M$ denote the set of smooth Riemannian metrics on M. We define the functional $\mathcal{K}_p \colon \mathcal{S}_M \to \mathbb{R}$ as

$$\mathcal{K}_p(g) = \int_M |R_g|_g^p \, d\mu_g$$

where R_g is the curvature tensor of g, $|\cdot|_g$ is the point-wise norm with respect to g, and $d\mu_g$ is the n-dimensional volume measure defined by g.

Picking a p

It turns out that for any other value of p than $n/2$, one may for any metric $g \in \mathcal{S}_M$ find a sequence of constants c_k such that the functional $\mathcal{K}_p(c_k g)$ tends to zero as k tends to infinity.
Thus from now on we consider the functional $\mathcal{K}$ defined by

$$\mathcal{K}(g) = \mathcal{K}_{n/2}(g) = \int_M |R_g|_g^{n/2} \, d\mu_g. \qquad (1)$$

We define a diffeomorphism invariant of M as the infimum over all metrics of (1), ie.

$$\mathcal{I}(M) = \inf_{g \in \mathcal{S}_M} \mathcal{K}(g),$$

and a metric g is called *optimal* if $\mathcal{K}(g) = \mathcal{I}(M)$, ie. if g is an absolute minimizer of (1).

Einstein metrics

A metric g on a manifold M is called *Einstein*, if at every point of M the Ricci tensor of g is a multiple of the metric, ie. $r = \lambda g$ for some function $\lambda \in C^\infty(M)$.

It is easy to see that the function λ is necessarily $\frac{1}{n}s$, where s is the scalar curvature and $n = \dim M$. In any dimension $n > 2$, the scalar curvature of an Einstein metric is constant.

Thus, in dimensions $n > 2$, a metric is Einstein if and only if its trace-free Ricci curvature $\mathring{r} = r - \frac{1}{n}s$ is identically zero.

The dimension 4

We now consider the case $n = 4$. We need to recall some facts from 4-dimensional geometry. Let (M, g) be an oriented Riemannian manifold. Denote by

$$\Lambda^2 = \Lambda^2 T^* M$$

the exterior square of the cotangent bundle of M; it is a rank 6 vector bundle over M. The Hodge star operator $\star \colon \Lambda^2 \to \Lambda^2$ is an involution of this vector bundle ($\star^2 = 1$), so there is a decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \qquad (2)$$

where $\Lambda^\pm$ is the ± 1 eigenspace of $\star$. Both are rank 3 vector bundles over M (they are interchanged if the orientation of M is reversed).

Splitting the curvature tensor

The curvature tensor R can, by an appropriate raising of indices, be considerered as a map $\Lambda^2 \to \Lambda^2$, and as such it splits according to (2):

$$R = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \mathring{r} & W_- + \frac{s}{12} \end{array} \right) \qquad (3)$$

where, by definition, $W_\pm$ are the trace-free parts of the appropriate blocks, $s = \mathrm{tr}_g\, r$ is the scalar curvature, and $\mathring{r} = r - \frac{s}{4}g$ is the trace-free part of the Ricci curvature.
In terms of this splitting, we get

$$|R|^2 = \frac{s^2}{24} + |W_+|^2 + |W_-|^2 + \frac{|\mathring{r}|^2}{2}. \qquad (4)$$

Topological invariants

There exists a 4-dimensional analogue of the Gauss-Bonnet formula. In terms of the splitting (3) it may be written

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu. \qquad (5)$$

where $\chi(M)$ is the *Euler characteristic* of M.

Signature

Another topological invariant is the *signature*: The intersection pairing $\smile : H^2(M) \times H^2(M) \to \mathbb{R}$ on de Rham cohomology defined by

$$[\varphi] \smile [\psi] = \int_M \varphi \wedge \psi$$

is

- non-degenerate, by Poincaré duality
- symmetric, since 2-forms commute with respect to the wedge product.

Signature (continued)

Thus we may find a basis for $H^2(M)$ in which the intersection pairing is represented by the diagonal matrix

$$\begin{pmatrix} I_{b_+} & 0 \\ 0 & -I_{b_-} \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix. The numbers $b_\pm = b_\pm(M)$ are topological invariants of M, and the signature $\tau(M)$ is defined to be the difference

$$\tau(M) = b_+(M) - b_-(M).$$

The Hirzebruch signature theorem states that the signature may be computed as

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W_+|^2 - |W_-|^2)\, d\mu. \qquad (6)$$

Einstein metrics are optimal

Combining the four-dimensional Gauss-Bonnet formula with the expression (4) for $|R|^2$, we see that

$$\int_M |R|^2 d\mu = 8\pi^2 \chi(M) + \int_M |\mathring{r}|^2 d\mu. \qquad (7)$$

This immediately has a couple of interesting consequences for compact 4-manifolds:

- $\mathcal{I}(M) \geq 8\pi^2 \chi(M)$.
- Any Einstein metric on M is optimal.
- If an Einstein metric exists, only Einstein metrics are optimal.

Then a natural question seems to be: Do all 4-manifolds admit Einstein metrics?

Remark

The above does not hold in any other dimension than 4: The standard metrics on S^{2k+1}, $S^{2k+1} \times S^3$ are Einstein, but do not minimize (1).

The Hitchin-Thorpe inequality

Combining the expressions (5) and (6) for the Euler characteristic and the signature, we obtain

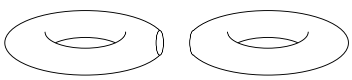
$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left(\frac{s^2}{24} + 2|W_\pm|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu.$$

In particular, we see that a necessary condition for M to admit an Einstein metric is that it satifies the Hitchin-Thorpe inequality

$$(2\chi \pm 3\tau)(M) \geq 0 \qquad (8)$$

It is easy to construct 4-manifolds not satisfying (8).

Connected sum



Let M_1 and M_2 be two compact connected oriented smooth n-manifolds.
The connected sum $M_1 \# M_2$ is the compact connected oriented smooth n-manifold obtained from the disjoint union of M_1 and M_2 by deleting a small open n-ball from each manifold, and identifying the two copies of S^{n-1} via a reflection.

If M_1 and M_2 are simply-connected, so is $M_1 \# M_2$, and has $b_\pm(M_1 \# M_2) = b_\pm(M_1) + b_\pm(M_2)$.

A family of counterexamples

Let $\mathbb{C}P^2$ denote the complex projective plane with its standard orientation, and $\overline{\mathbb{C}P^2}$ the same manifold, but with the opposite orientation.

The iterated connected sum

$$j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2} = \underbrace{\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2}_{j} \# \underbrace{\overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}}_{k}$$

is a simply connected compact 4-manifold with $b_+ = j$, $b_- = k$ and $\chi = 2 + j + k$. In particular, the quantity

$$(2\chi + 3\tau)(j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}) = 4 + 5j - k$$

is negative for sufficiently large k (with respect to j), so these manifolds do not admit Einstein metrics.

Another source of optimal metrics

While not all 4-manifolds carry Einstein metrics, we have not yet answered the question whether they all carry optimal metrics. Rewriting the functional $\mathcal{K}$ using (5) and (6) we obtain

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu \qquad (9)$$

Another class of metrics is the scalar-flat anti-self-dual ones. If M is a smooth 4-manifold, a Riemannian metric g on M is called

- *scalar-flat*, if it satifies $s = 0$
- *anti-self-dual*, if its self-dual Weyl curvature W_+ is identically 0.

A metric satisfying both of these conditions will be called scalar-flat anti-self-dual (SFASD).

SFASD metrics are optimal

In case a SFASD exists, we know the value of $\mathcal{I}(M)$ by the formula

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu$$

so that

$$\mathcal{I}(M) = -8\pi^2(\chi + 3\tau)(M) \qquad (10)$$

Similar to the case of Einstein metrics, we see immediately that a SFASD metric on M is optimal, and if M admits such a metric these are the only optimal metrics. Also, the reverse Hitchin-Thorpe inequality is a topological obstruction to admitting a SFASD metric:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \leq 0.$$

Manifolds admitting SFASD metrics

Another topological obstruction to admitting a SFASD metric is the following

Theorem

Let M be a simply connected smooth compact 4-manifold. If M admits a SFASD metric, then

- *M is homeomorphic to $k\overline{\mathbb{C}P^2}$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

This should be compared with the following list of smooth manifolds for which SFASD metrics are known:

Theorem

The following smooth 4-manifolds actually admit SFASD metrics:

- *$k\overline{\mathbb{C}P^2}$ (with its standard smooth structure), for $k \geq 6$;*
- *$\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, for $k \geq 14$; and*
- *$K3$.*

Anorexic sequences

A sequence g_j of metrics on M will be called *anorexic*, if $\int s^2 d\mu \to 0$ and $\int |W_+|^2 d\mu \to 0$. If M admits an anorexic sequence, the value of $\mathcal{I}(M)$ is $-8\pi^2(\chi + 3\tau)(M)$, and any optimal metric is SFASD.

Theorem

The following smooth simply connected manifolds admit anorexic sequences of metrics

- $4\overline{\mathbb{C}P^2}$
- $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$
- *$j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $j \geq 2$, $k \geq 9j$.*

However, these manifolds do not admit SFASD metrics, and hence do not admit optimal metrics.

Final remarks

- Existence of optimal metrics depends strictly on diffeotype, not only homeotype:
  - For any odd $j \geq 1$ and $k \geq 5j + 4$, the topological manifold $j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ admits infinitely many distinct differentiable structures for which no optimal metric exists.
  - However, for $j = 1$ and $k \geq 14$, the manifold carries a *standard* smooth structure which *do* admit an optimal metric.
  - The topological manifold $7\mathbb{C}P^2 \# 37\overline{\mathbb{C}P^2}$ admits one smooth structure for which an Einstein metric exists, and another smooth structure for which no Einstein metric exists.
- Existence of SFASD metrics on $5\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $10 \leq k \leq 13$ is still unknown.

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A quote

[...] imagine a curvaceous young 4-manifold who, bedazzled by the glamorous starlets with optimal metrics she has been reading about in the tabloids, suddenly decides to go on a starvation diet to get rid of all that unwanted curvature. If she has the wrong body type, this misguided procedure will be dangerous to her health, and she will merely succeed in putting herself in the hospital.

—Claude LeBrun