SURFACE AREA AND VOLUME OF A TORUS

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1 Abstract

This is a short calculation of the surface area and volume of a 2-dimensional torus embedded in $\mathbb{R}^3$, with inner radius b and outer radius c.

2 Surfacearea and volume and volume of a torus

The upper arc of the generating circle for the torus is

$$f_1(x) = A + \sqrt{a^2 - x^2} \quad a, A > 0,$$

where $b = A - a$ is the inner radius of the torus. That is A is the distance from the center of the torus to the center of the generating circle, and a is the radius of the generating circle. The lower arc of the generating circle is in the same way given by

$$f_2(x) = A - \sqrt{a^2 - x^2}.$$

Rotation of f_1 about the x-axis will give a figure shaped like a french cheese. If you from this subract f_2 rotated about the x-axis you get a torus.

Volume

The volume is calculated by using the formula for the volume of a surface of revolution, generated by the function $f(x)$ for $x \in [a, b]$

$$V = \pi \int_a^b f(x)^2 \, dx.$$

So

$$\begin{aligned}
V =& \pi \int_{-a}^{a} (A + \sqrt{a^2 - x^2})^2 - (A - \sqrt{a^2 - x^2})^2 dx \\
=& \pi \Big[Aa^2 \sin^{-1}\left(\frac{x}{a}\right) + Ax\sqrt{a^2 - x^2} - \frac{x^3}{3} + (A^2 + a^2)x \\
& + Aa^2 \sin^{-1}\left(\frac{x}{a}\right) + Ax\sqrt{a^2 - x^2} + \frac{x^3}{3} - (A^2 + a^2)x \Big]_{-a}^{a} \\
=& \pi \Big[2Aa^2 \sin^{-1}\left(\frac{x}{a}\right) + 2Ax\sqrt{a^2 - x^2} \Big]_{-a}^{a} \\
=& 2\pi^2 Aa^2.
\end{aligned}$$

If b is the inner radius of the torus, and c is the outer radius of the torus, then $A = \frac{b+c}{2}$ and $a = \frac{c-b}{2}$ and the volume will be

$$V = \frac{\pi^2}{4}(b + c)(c - b)^2.$$

Surface area

The surface area is calculated by using the formula for the surface area of a surface of revolution around the x-axis, described by the function $f(x)$ for $x \in [a; b]$

$$A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} \, dx.$$

This forumula needs the derivatives of f_1 and f_2.

$$f_1'(x) = \frac{-x}{\sqrt{a^2 - x^2}} \qquad f_2' = \frac{x}{\sqrt{a^2 - x^2}}.$$

$$\begin{aligned} A &= 2\pi \int_{-a}^{a} f_1(x)\sqrt{1 + f_1'(x)^2} + f_2(x)\sqrt{1 + f_2'(x)^2} dx \\ &= 4\pi Aa \left[\arctan\left(\frac{x}{\sqrt{a^2 - x^2}} \right) \right]_{-a}^{a} \\ &= 4\pi^2 Aa \\ &= \pi(b + c)(c - b). \end{aligned}$$

This is just brute force calculations. You could also use

$$x(u, v) = (Aa\cos u \cos v, Aa\cos u \sin v, a\sin u) \quad (u, v) \in (0, 2\pi) \times (0, 2\pi) = U$$

as a parametrization of the same torus. Then you could use the formulas for surface areas of 2-dimensional surface, R, in $\mathbb{R}^3$:

$$A(R) = \int_{x^{-1}(R)} |x_u \wedge x_v| \; dudv = \int_{x^{-1}(R)} \sqrt{EG - F^2} \, dudv,$$

where $E, G, F : U \to \mathbb{R}$ are the coefficients to the first fundamental form, like in [do Carmo(1976)], $R \subset U'$ is a compact set, and (U, x) is a chart covering the surface of which you want to calculate the area.[1] Furthermore x_u and x_v are shorthand notation for the partial derivatives of x.

$x : U \to U' \supset R$ covers the whole torus except to circles. But they are of dimension 1 and the surface integral is just zero on a 1-dimensional curve, so it doesn't change the totale area.

$$\begin{aligned} E &= \langle x_u, x_u \rangle = r^2 \\ F &= \langle x_u, x_v \rangle = 0 \\ G &= \langle x_v, x_v \rangle = (a\cos u - A)^2 \end{aligned}$$

If $(u, v) \in (\varepsilon, 2\pi - \varepsilon) \times (\varepsilon, 2\pi - \varepsilon)$ and

$$R_\varepsilon = \overline{x((\varepsilon, 2\pi - \varepsilon) \times (\varepsilon, 2\pi - \varepsilon))}$$

then $A(R_\varepsilon) \to A(R)$ as $\varepsilon \to 0$.

$$\begin{aligned} \int_{\varepsilon}^{2\pi-\varepsilon} \int_{\varepsilon}^{2\pi-\varepsilon} a^2 \cos u + Aa \, dudv &= \int_{\varepsilon}^{2\pi-\varepsilon} a(a\cos u + A)2(\pi - \varepsilon) \, du \\ &= 4(\pi - \varepsilon)^2 aA + 2(\pi - \varepsilon)a \int_{\varepsilon}^{2\pi-\varepsilon} a\cos u \, du \\ &= 4(\pi - \varepsilon)^2 aA + 2(\pi - \varepsilon)a^2(\sin(2\pi - \varepsilon) - \sin(\varepsilon)) \\ &\to 4\pi Aa \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Bibliography

[do Carmo(1976)] Manfredo do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice Hall, 1976.

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[1] In Riemannian geometry the charts are normally given as homeomorphisms from the surface to a subset of a Eucledian space, but here x is a homeomorphism from $\mathbb{R}^2$ to the surface