

DETERMINANT OF AN
ENDOMORPHISM

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1 Abstract

In this note we define the determinant of an endomorphism, between finite dimensional vectorspaces, and we show that the definition is well defined.

2 Determinant of an endomorphism

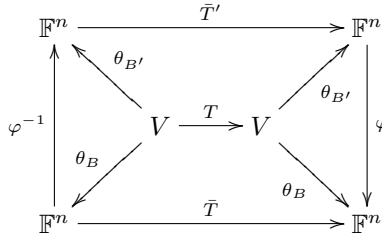
First of all, an endomorphism between finite dimensional vectorspaces is a linear transformation from a vector space to itself, i.e. $T : V \rightarrow V$, where T is a linear transformation and V a finite dimensional vector space. The matrix representation of T with respect to a basis is a quadratic matrix, and we define the determinant of this matrix, to be the determinant of the endomorphism.

We then need to see that this is well defined. The following proposition gives the key ingredient.

Proposition 2.1. *Let $T : V \rightarrow V$ be a linear transformation, and V is a finite dimensional vector space, and let B and B' be two ordered bases for V . Then a matrix representation for T with respect to B, B is similar with the matrix representation with respect to B', B' .*

Proof. Two matrices, A, B are similar if there exists an invertible matrix C , such that $A = C^{-1}BC$.

The following commutative diagram is the essential ingredient in the proof.



Where \bar{T}' and \bar{T} are the matrix representations for V in terms of the bases B' and B , and the maps φ and φ^{-1} are the basis transition maps. Furthermore the maps $\theta_{B'}$ and θ_B are the coordinate maps. Then $\bar{T} = \varphi^{-1} \circ \bar{T}' \circ \varphi$.

The coordinate maps are isomorphisms, so the composition $\varphi = \theta_B \circ \theta_{B'}$ is an isomorphism. This implies that φ is invertible as a matrix. Likewise for φ^{-1} . This means that we have found an invertible matrix that makes \bar{T} and \bar{T}' similar. \square

The only thing we need know is that similar matrices have the same determinant.

Theorem 2.2. *Let $A, B, C \in Mat_n(\mathbb{F})$ and C is invertible, and assume $A = C^{-1}BC$, then $\det(A) = \det(B)$.*

Proof.

$$\begin{aligned}
 \det(A) &= \det(C^{-1}BC) = \det(C^{-1}) \det(B) \det(C) \\
 &= \det(C)^{-1} \det(B) \det(C) \\
 &= \det(B).
 \end{aligned}$$

\square

This proves that the definition of the determinant of an endomorphism is well defined.