$\mathcal{F}$ presheaf on X.

$$\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U) = \{ s_p \mid s \in \mathcal{F}(U),\ p \in U \} \quad \text{stalk of } \mathcal{F} \text{ at } p.$$

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ gives $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ s.t.

$$\varphi_p(s_p) = \varphi_U(s)_p \qquad \forall s \in \mathcal{F}(U),\ p \in U.$$

<u>Prop</u> $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of <u>sheaves</u>. Then φ isomorphism $\iff \varphi_p : \mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$ iso. $\forall p \in X$.

<u>Proof</u> $\Rightarrow$: clear.

$\Leftarrow$: Enough to show $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ iso. $\forall U \subseteq X$ open.

<u>φ_U injective</u>: If $\varphi_U(s) = 0 \in \mathcal{G}(U)$ then $\varphi_p(s_p) = \varphi_U(s)_p = 0$ $\forall p \in U$

$\Rightarrow s_p = 0$ $\forall p$ $\Rightarrow s = 0 \in \mathcal{F}(U)$.

<u>φ_U surjective</u>: Let $t \in \mathcal{G}(U)$.

$\in U$: φ_p surjective $\Rightarrow$ $t_p = \varphi_p\big(s(p)_p \big) \in \mathcal{G}_p$

for some $s(p) \in \mathcal{F}(V_p)$, $p \in V_p \subseteq U$.

Now $t_p = \varphi_{V_p}(s(p))_p$. Make V_p smaller so that $t|_{V_p} = \varphi_{V_p}(s(p))$.

$$\varphi\big(s(p)|_{V_p \cap V_q} \big) = t|_{V_p \cap V_q} = \varphi\big(s(q)|_{V_p \cap V_q} \big)$$

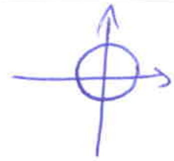
$$\Rightarrow s(p)|_{V_p \cap V_q} = s(q)_{V_p \cap V_q} \quad \forall p, q.$$

$\therefore$ $\exists !\ s \in \mathcal{F}(U)$ s.t. $s|_{V_p} = s(p) \in \mathcal{F}(V_p)$ $\forall p$.

$$\varphi_U(s)|_{V_p} = \varphi_{V_p}(s(p)) = t|_{V_p} \quad \Rightarrow \varphi_U(s) = t \in \mathcal{G}(U).$$

$\square$

<u>Remark</u> $\mathcal{F}$ sheaf on X, $U \subseteq X$ open. Then we have sheaf $\mathcal{F}|_U$ on U defined by $\Gamma(V, \mathcal{F}|_U) = \Gamma(V, \mathcal{F})$.



Example $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

Define sheaves $\mathcal{F}$, $\mathcal{G}$ on S^1 as follows.

$\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ locally const.}\}$

$\mathcal{G}(U) = \{f : U \to \mathbb{R} \mid f\ C^\infty,\ \frac{\partial f}{\partial t} = 1\}$

If $U \subsetneq S^1$ then $\mathcal{F}|_U \cong \mathcal{G}|_U$, $f(x,y) \mapsto f(x,y) + \arg(x,y)$.

In particular $\mathcal{F}_p \cong \mathcal{G}_p \ \forall p \in S^1$.

BUT $\mathcal{F} \not\cong \mathcal{G}$. In fact $\mathcal{F}(S^1) = \mathbb{R}$, $\mathcal{G}(S^1) = \emptyset$.

Problem: $\nexists$ morphism $\mathcal{F} \to \mathcal{G}$.

Sheafification

$\mathcal{F}$ presheaf on X. Define a sheaf $\mathcal{F}^+$ as follows.

If $U \subseteq X$ open, set $\mathcal{F}^+(U)$ = set of functions $s : U \longrightarrow \coprod_{p \in U} \mathcal{F}_p$ s.t.

1) $s(p) \in \mathcal{F}_p$.
2) $\forall p \in U\ \exists$ open V, $p \in V \subseteq U$ and $t \in \mathcal{F}(V)$ s.t. $s(q) = t_q \ \forall q \in V$.

Def. morphism $\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$: If $t \in \mathcal{F}(U)$ then $\theta_U(t) = [p \mapsto t_p] \in \mathcal{F}^+(U)$.

Exercise $\theta_p : \mathcal{F}_p \longrightarrow \mathcal{F}^+_p$ isomorphism $\forall p \in X$.

Prop $\mathcal{F}$ presheaf, $\mathcal{G}$ sheaf, $\varphi : \mathcal{F} \to \mathcal{G}$ morphism.
Then $\exists!\ \varphi^+ : \mathcal{F}^+ \longrightarrow \mathcal{G}$ s.t. $\varphi = \varphi^+ \circ \theta$.

Proof Let $s \in \mathcal{F}^+(U)$, i.e. $s : U \to \coprod \mathcal{F}_p$.

For $p \in U$, choose $t(p) \in \mathcal{F}(V_p)$, $p \in V_p \subseteq U$ s.t. $s(q) = t(p)_q \ \forall q \in V_p$.

Set $\tau(p) = \varphi_{V_p}(t(p)) \in \mathcal{G}(V_p)$.

If $q \in V_p$ then $\tau(p)_q = \varphi(t(p))_q = \varphi_q(t(p)_q) = \varphi_q(s(q)) \in \mathcal{G}_q$.

$\Rightarrow \tau(p_1) = \tau(p_2)$ on $V_{p_1} \cap V_{p_2} \Rightarrow$ the $\tau(p)$ glue to $\tau \in \mathcal{G}(U)$.

Set $\varphi^+_U(s) = \tau$.

$\square$

<u>Remark</u> $\mathcal{F}^+$ is uniquely determined by the universal property of this proposition (up to unique isomorphism.).

<u>Def</u> $\varphi : \mathcal{F}' \to \mathcal{F}$ morphism of sheaves. φ is <u>injective</u> if φ_U injective $\forall U$ open.

(can think of $\mathcal{F}'$ as subsheaf of $\mathcal{F}$, i.e. $\mathcal{F}'(U) \subseteq \mathcal{F}(U)$ $\forall U$.)

<u>Exercise</u> φ is injective $\Leftrightarrow$ $\varphi_p : \mathcal{F}'_p \to \mathcal{F}_p$ injective $\forall p \in X$.

<u>Consequence</u>: If $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of presheaves s.t. $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective $\forall U$ then $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ is injective.

(Enough to check that $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ injective $\forall p$.
If $\varphi_p(s_p) = 0 \in \mathcal{G}_p$ for some $s \in \mathcal{F}(U)$, $p \in U$ then
$\varphi_U(s)_p = 0 \Rightarrow \varphi_U(s)|_V = 0 \Rightarrow \varphi_V(s|_V) = 0 \Rightarrow s|_V = 0 \Rightarrow s_p = 0$.)

<u>In particular</u>: If a pre-sheaf $\mathcal{F}$ is a subpresheaf of a <u>sheaf</u> $\mathcal{G}$ then $\mathcal{F}^+ \subseteq \mathcal{G}$. (In fact $\mathcal{F}^+ \subseteq \mathcal{G}^+ = \mathcal{G}$.)

<u>Def</u> $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves of abelian groups.

$\ker(\varphi)$ = the <u>sheaf</u> $U \mapsto \ker(\varphi_U) \subseteq \mathcal{F}(U)$.

$\mathrm{Im}(\varphi)$ = the <u>sheafification</u> of the <u>presheaf</u> $U \mapsto \mathrm{Im}(\varphi_U) \subseteq \mathcal{G}(U)$.

<u>Note</u> $\ker(\varphi) \subseteq \mathcal{F}$ and $\mathrm{Im}(\varphi) \subseteq \mathcal{G}$ subsheaves.

φ injective $\Leftrightarrow$ $\ker(\varphi) = 0$.

<u>Def</u> φ is <u>surjective</u> if $\mathrm{Im}(\varphi) = \mathcal{G}$.

<u>Warning</u> φ surjective $\not\Rightarrow$ φ_U surjective!

<u>Exercise</u> $\mathrm{Im}(\varphi)_p = \varphi_p(\mathcal{F}_p) \subseteq \mathcal{G}_p$

$\therefore$ φ surjective $\Leftrightarrow$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ surjective $\forall p \in X$.

<u>Def</u> $\mathcal{F}' \subseteq \mathcal{F}$ subsheaf. Set $\mathcal{F}/\mathcal{F}' = \left[U \mapsto \mathcal{F}(U)/\mathcal{F}'(U) \right]^+$.

Note: Have surjective morphism $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}'$ and $\ker(\mathcal{F} \to \mathcal{F}/\mathcal{F}') = \mathcal{F}'$.

<u>Example</u> X variety, $Y \subseteq X$ closed subvariety.

For $U \subseteq X$ open, set $I_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f(y) = 0 \ \forall y \in Y \cap U \}$.

Then $I_Y \subseteq \mathcal{O}_X$ subsheaf.

$$\mathcal{O}_X(U) / I_Y(U) = \left\{ \begin{array}{l} \text{regular fcns. } f: U \cap Y \to k \text{ which can be extended} \\ \text{to all of } U \end{array} \right\}.$$

We have: $\mathcal{O}_X / I_Y = \mathcal{O}_Y$ structure sheaf on Y.

Reason: Every regular function on Y can be extended to X <u>locally</u>.